Applied Mathematics

Class XI

CENTRAL BOARD OF SECONDARY EDUCATION

Shiksha Sadan, 17, Institutional Area,
Rouse Avenue, New Delhi-110 002

Dedicated to:

Sh. Pramod Kumar T.K.

(15th April 1971 to 25th May 2021)

Joint Secretary (Academics)
Central Board of Secondary Education

You will always be in our memories...

Applied Mathematics

Class XI

Student Support Material

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भारत का संविधान

उद्देशिका

हम, भारत के लोग, भारत को एक सम्पूर्ण [1]प्रभुत्व-संपन्न समाजवादी पंथनिरपेक्ष लोकतंत्रात्मक गणराज्य बनाने के लिए, तथा उसके समस्त नागरिकों को:

सामाजिक, आर्थिक और राजनैतिक न्याय,
विचार, अभिव्यक्ति, विश्वास, धर्म
और उपासना की स्वतंत्रता,
प्रतिष्ठा और अवसर की समता
प्राप्त कराने के लिए
तथा उन सब में व्यक्ति की गरिमा
[2]और राष्ट्र की एकता और अखंडता
सुनिश्चित करने वाली बंधुता बढ़ाने के लिए

दृढ़संकल्प होकर अपनी इस संविधान सभा में आज तारीख 26 नवम्बर, 1949 ई० को एतद्द्वारा इस संविधान को अंगीकृत, अधिनियमित और आत्मार्पित करते हैं।

1. संविधान (बयालीसवां संशोधन) अधिनियम, 1976 की धारा 2 द्वारा (3.1.1977) से "प्रभुत्व-संपन्न लोकतंत्रात्मक गणराज्य" के स्थान पर प्रतिस्थापित।
2. संविधान (बयालीसवां संशोधन) अधिनियम, 1976 की धारा 2 द्वारा (3.1.1977) से "राष्ट्र की एकता" के स्थान पर प्रतिस्थापित।

भाग 4 क

मूल कर्तव्य

51 क. मूल कर्तव्य – भारत के प्रत्येक नागरिक का यह कर्तव्य होगा कि वह –

(क) संविधान का पालन करे और उसके आदर्शों, संस्थाओं, राष्ट्रध्वज और राष्ट्रगान का आदर करे;

(ख) स्वतंत्रता के लिए हमारे राष्ट्रीय आंदोलन को प्रेरित करने वाले उच्च आदर्शों को हृदय में संजोए रखे और उनका पालन करे;

(ग) भारत की प्रभुता, एकता और अखंडता की रक्षा करे और उसे अक्षुण्ण रखे;

(घ) देश की रक्षा करे और आह्वान किए जाने पर राष्ट्र की सेवा करे;

(ङ) भारत के सभी लोगों में समरसता और समान भ्रातृत्व की भावना का निर्माण करे जो धर्म, भाषा और प्रदेश या वर्ग पर आधारित सभी भेदभाव से परे हों, ऐसी प्रथाओं का त्याग करे जो स्त्रियों के सम्मान के विरुद्ध हैं;

(च) हमारी सामासिक संस्कृति की गौरवशाली परंपरा का महत्त्व समझे और उसका परिरक्षण करे;

(छ) प्राकृतिक पर्यावरण की जिसके अंतर्गत वन, झील, नदी, और वन्य जीव हैं, रक्षा करे और उसका संवर्धन करे तथा प्राणी मात्र के प्रति दयाभाव रखे;

(ज) वैज्ञानिक दृष्टिकोण, मानववाद और ज्ञानार्जन तथा सुधार की भावना का विकास करे;

(झ) सार्वजनिक संपत्ति को सुरक्षित रखे और हिंसा से दूर रहे;

(ञ) व्यक्तिगत और सामूहिक गतिविधियों के सभी क्षेत्रों में उत्कर्ष की ओर बढ़ने का सतत प्रयास करे जिससे राष्ट्र निरंतर बढ़ते हुए प्रयत्न और उपलब्धि की नई उंचाइयों को छू ले;

[1](ट) यदि माता-पिता या संरक्षक है, छह वर्ष से चौदह वर्ष तक की आयु वाले अपने, यथास्थिति, बालक या प्रतिपाल्य के लिये शिक्षा के अवसर प्रदान करे।

1. संविधान (छयासीवां संशोधन) अधिनियम, 2002 की धारा 4 द्वारा प्रतिस्थापित।

THE CONSTITUTION OF INDIA

PREAMBLE

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a [1]**[SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC]** and to secure to all its citizens :

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the[2] [unity and integrity of the Nation];

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949, do **HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.**

1. Subs, by the Constitution (Forty-Second Amendment) Act. 1976, sec. 2, for "Sovereign Democratic Republic" (w.e.f. 3.1.1977)
2. Subs, by the Constitution (Forty-Second Amendment) Act. 1976, sec. 2, for "unity of the Nation" (w.e.f. 3.1.1977)

THE CONSTITUTION OF INDIA

Chapter IV A

FUNDAMENTAL DUTIES

ARTICLE 51A

Fundamental Duties - It shall be the duty of every citizen of India-

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forests, lakes, rivers, wild life and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;

[1](k) who is a parent or guardian to provide opportunities for education to his/her child or, as the case may be, ward between age of six and forteen years.

1. Ins. by the constitution (Eighty - Sixth Amendment) Act, 2002 S.4 (w.e.f. 12.12.2002)

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Note to the user

The technological, economic, and social changes of the last few decades have redefined the scope and relevance of important mathematical ideas in work and in everyday life. With exponential rise in applications of Mathematics influencing almost every aspect of human lives, it necessitates preparing mathematically skilled workforce who can make informed decisions in their professional and personal lives. India has rich and diverse landscape of learning contexts that requires effective mathematical solutions. Such mathematically illuminated contexts can contribute to the development of mathematics curriculum mapping the country's need and spirit. Moreover Mathematics skills are very much essential in all the fields be it art, literature or social sciences.

The present course in Applied Mathematics is application centric and problem driven. Key mathematical ideas are developed within and around the contexts that have potential applications in practical life. The course aims to develop new perspectives about what Mathematics is and how it can be used. The course is divided into different units each focusing on numerical, algebraic or statistical applications. The scope of the course ranges from mathematical applications in the field of financial and services (including Business and Economics), Pharmaceutical industry, Hospitality and Entertainment industry, Advertisement and Marketing sector, Healthcare Services including Counseling Services, Agricultural Sector, Government-aided programs such as resources distribution, Traffic Management Systems including Air Traffic Management, Infrastructure Development, Archeological Studies and studies on Historiography, real time data analysis and many more. Core mathematical ideas are presented keeping in mind these contexts.

The course aims to develop among learners the habit of mathematical thinking by reasoning quantitatively; constructing viable mathematical arguments; constructing new mathematical knowledge using problem solving and learning to model with Mathematics.

This student support material is divided into different units including a unit on use of spreadsheet for mathematical applications. Sequencing of units is done in a natural order staring from Quantification and Numerical Applications, Algebraic Thinking, Mathematical Reasoning, Calculus, Probability, Descriptive Statistics, Financial Mathematics and Coordinate Geometry.

Each unit begins with a starting position and divided into a number of sections having discrete exercises. Efforts are done to develop concepts around meaningful applications to foster inquisitiveness in learners. Teachers are encouraged to extend/adapt these examples to initiate investigations and stimulating discourse in the classrooms.

It may be kept in mind that this should not be considered as a text book and teachers should avoid teaching the content as given in this book. It only gives some directions to approach various topics. Teachers are advised to use multiple examples from sources such as newspapers, magazines and other online resources to help students reconcile different approaches of problem solving and refining mathematical arguments.

We hope that this book will provide students a real sense of application of Mathematics in different spheres of work and life. It is an invitation to reinvent mathematics and how it can empower us to make more logical and informed decisions in lives.

We look forward to the suggestions from teachers, students and all learning community to make this book more realistic for Indian classrooms.

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Chapter-I

Numbers and Quantification

Learning Objectives :

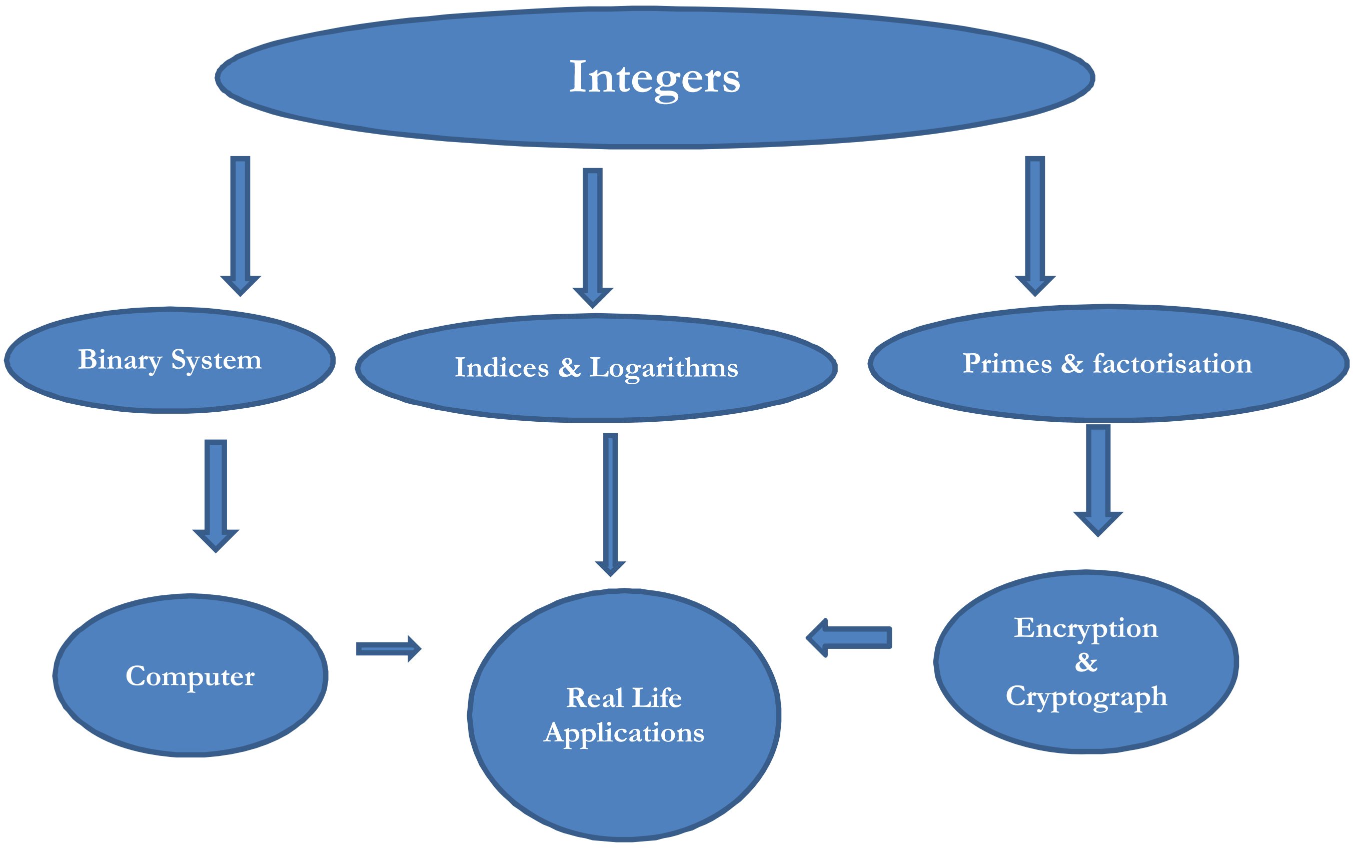
Students will be able to:

- identify prime numbers;
- explain the importance of prime numbers particularly in encryption;
- write numbers in binary system;
- explain basic notions of complex numbers and why are they introduced;
- use logarithms in different applications;
- above ideas in the day to day life.

Previous Knowledge:

The ideas and concepts are self contained and based on Secondary Mathematics.

Concept Map



1.1 Introduction:

In this modern age of computers, primes play an important role to make our communications secure. So understanding primes are very important. In Section 1.1., we give an introduction to primes and its properties and give its application to encryption in 1.2. The concept of binary numbers and writing a number in binary and converting a number in binary to decimal is presented in 1.3. The Section 1.4 introduces the concept of complex numbers with some basic properties. In Sections 1.5, 1.6 and 1.7. logarithms, its properties and applications to real life problems are introduced. Finally in Section 1.8 we give a number of word problems which we come across in everyday life.

1.2 Prime Numbers

We say integer a divides b, and write $a \mid b$ in symbols, if $b = ac$ with $c \in Z$. For example $9 \mid 27$. If $a \mid b$, we say a is a factor or a is divisor of b. And we also say b is a multiple of a. It is easy to see that 1 and −1 are divisors of any integer and every non-zero integer is a divisor of 0. We write $a \nmid b$ if a does not divide b. For example $5 \nmid 13$.

Prime numbers are the building blocks of number system. A positive integer $p > 1$ is called a prime if 1 and p are the only positive divisors of p. For example, 2, 3, 5, 7, 11, 13, ... are primes. We say a positive integer $n > 1$ is composite if it is not a prime. Thus a composite number $n > 1$ has a positive divisor different from 1 and n itself. For example, 4, 6, 8, 9, 10, ... are composites. The number 1 is neither a prime nor a composite.

One of the important properties of positive integers for which the primes play an important role is The Fundamental Theorem of Arithmetic. This theorem states that every positive integer $n > 1$ can be written as product of primes and it is unique upto order of primes. Hence every $n \in N$ can be written uniquely in the form

$$n = P_1^{a_1} P_2^{a_2} P_3^{a_3} .....P_r^{a_r}$$

where $p_1 < p_2 < ..... < p_r$ are distinct primes and $a_1, a_2, ..., a_r$ are positive integers. This is called the unique factorization of n. For example

$$10 = 2 \cdot 5$$

$$36 = 2^2 \cdot 3^2$$

$$105 = 3 \cdot 5 \cdot 7$$

$$104568 = 2^3 \cdot 3 \cdot 4357.$$

For a prime p, the factorization of p is just $p = p$. Now you can answer why 1 is not considered a prime. If 1 is a prime, then

$$10 = 1 \cdot 2 \cdot 5 = 1^2 \cdot 2 \cdot 5 = 1^3 \cdot 2 \cdot 5 = ... = 1^r \cdot 2 \cdot 5$$

for any $r > 1$, thereby giving a number of factorization of 10, thereby violating the Fundamental Theorem of Arithmetic. Also 1 cannot be a considered a composite number. (Why?)

A consequence of the Fundamental Theorem of Arithmetic is that every $n > 1$ has a prime divisor p. Clearly n itself is the unique prime divisor p if n is a prime and $1 < p < n$ if n is composite. This consequence implies one of the first theorem in Mathematics, which is well-known as Euclid's Theorem.

Theorem 1.1.1. Euclid's Theorem: There are infinitely many prime numbers.

Proof. Suppose there are finitely many prime numbers, viz., p_1, p_2, ... , p_r. Consider the number.

$$N = p_1 p_2 \ldots\ldots p_r + 1$$

Clearly $N > 1$. Hence by the Fundamental Theorem of Arithmetic, it has prime divisor p. Then $p = p_i$ for some $1 \leq i \leq r$. However none of the primes $p_i \mid N$ for each $1 \leq i \leq r$.

Also $N \neq p_i$ for any i. This is a contradiction which proves that there are infinitely many primes.

Though Euclid's Theorem tells us that there are infinitely many primes, finding large numbers is a challenge. In fact the largest known prime number as of today is the 24862048 digit prime

$$2^{82589933} - 1$$

which was discovered in 2018. This is a special kind of primes called Mersenne Primes.

One of the ways to check whether a given number is a prime is the well-known Sieve of Erasthosthenes. This works on the principle that $n > 1$ is composite if has a prime divisor $p \leq \sqrt{n}$. We illustrate this Sieve and find all primes upto 100. We list all positive integers upto 100. A numbers $1 < n \leq 100$ is a composite if n has a prime divisor $\leq \sqrt{n} = \sqrt{100} = 10$.

The primes upto 10 are 2, 3, 5, 7. We start by crossing 1. Since 2 is a prime, we circle 2 and strike off all numbers divisible by 2.

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~
11	~~12~~	13	~~14~~	15	~~16~~	17	~~18~~	19	~~20~~
21	~~22~~	23	~~24~~	25	~~26~~	27	~~28~~	29	~~30~~
31	~~32~~	33	~~34~~	35	~~36~~	37	~~38~~	39	~~40~~
41	~~42~~	43	~~44~~	45	~~46~~	47	~~48~~	49	~~50~~
51	~~52~~	53	~~54~~	55	~~56~~	57	~~58~~	59	~~60~~
61	~~62~~	63	~~64~~	65	~~66~~	67	~~68~~	69	~~70~~
71	~~72~~	73	~~74~~	75	~~76~~	77	~~78~~	79	~~80~~
81	~~82~~	83	~~84~~	85	~~86~~	87	~~88~~	89	~~90~~
91	~~92~~	93	~~94~~	95	~~96~~	97	~~98~~	99	~~100~~

3 is the first number among the remaining ones. We circle 3 now and strike off all numbers divisible by 3. 5 is the first among the remaining numbers which we circle and strike off all numbers divisible by 5. Next 7 is the first number left which we circle now and strike off all numbers divisible by 7.

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

The remaining numbers are all primes. In fact all the primes upto 100 are

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

which are 25 in number. This method to find primes is not suitable when the numbers are very large.

Suggested Project: Find all prime numbers upto 10000 by using the Sieve of Erasthosthenes.

Srinivasa Ramanujan is one of Indian genius who is very much well-known for his contributions to Mathematics, particularly prime numbers. Given n ≥ 1, we define the n-th Ramanujan Prime to be the least positive integer Rn such that there are at least n primes between $\pi(\frac{x}{2})$ and $\pi x \forall x \geq R_n$. It turns out that Rn are all primes. The first ten Ramanujan primes are

$R_1 = 2, R_2 = 11, R_3 = 17, R_4 = 29, R_5 = 41, R_6 = 47, R_7 = 59, R_8 = 67, R_9 = 71, R_{10} = 97.$

1.3 Why prime numbers are important?: Encryptions using Prime Numbers

Primes are one of the most useful numbers nowadays as they have lots of applications in the current digital world. In fact, prime numbers are used to make our online communications secure. In this section, we will explain how prime numbers are used in Cryptography.

Cryptography is the science of using mathematics to encrypt and decrypt data. It enable us to store sensitive information or transmit it across insecure networks (like the Internet) so that it cannot be read by anyone except the intended recipient. Cryptanalysis is the science of analyzing and breaking secure communication. It involves an interesting combination of analytical reasoning, application of mathematical tools, pattern finding, patience, determination, and luck. Basically Cryptanalysts are attackers and Cryptographers are defenders. Cryptology is the science which involves both Cryptography and Cryptanalysis.

In Cryptography, we have the notion of Encryption and Decryption. Encryption is the method of disguising plaintext (or the message in Data format which can be read and understood without any special measures) in such a way as to hide its substance. Encrypting plaintext results in unreadable gibberish called Cipher text. Encryption ensures that information is hidden from anyone for whom it is not intended, even those who can see the encrypted data. Decryption is the process of reverting cipher text to its original plaintext.

Cryptography works by using a cryptographic algorithm which is a mathematical function used in encryption and decryption process. It works in combination with a key (a word, number or phrase) to encrypt the plaintext.

Same plaintext encrypts to different ciphertext with different keys. The security of encrypted data depends entirely on two things. The strength of the cryptographic algorithm and the secrecy of the key. A Cryptosystem is a cryptographic algorithm plus all possible keys and all the protocols that make it work. Following are the requirements for a good cryptosystem.

(a) Authentication: Provides the assurance of some ones identity.

(b) Confidentiality : Protects against disclosure to unauthorized identities.

(c) Non-Repudiation: Protects against communications originator to later deny it.

(d) Integrity: Protects from unauthorized data alteration.

Some of the Cryptosystems are RSA, Diffie-Hellman, ElGamal, Elliptic Curve Cryptosystems. RSA which was invented in 1978 is one of the most popular and widely used cryptosystem. It is named after its inventors Ron Rivest, Adi Shamir and Richard Adleman. This cryptosystem is based on the property of primes. We will now explain the RSA Cryptosystem and illustrate how the prime numbers are used. For that we need some basics.

Definition: A positive integer d is the greatest common divisor or highest common factor of two numbers a and b if d is the largest positive common divisor of a and b, i.e.,

$d \mid a$ and $d \mid b$,

If $c \mid a$ and $c \mid b$, then $c \leq d$.

We write d = gcd(a, b) or simply d = (a, b) if d is the greatest common divisor of a and b. For example

gcd (10, 25) = 5

gcd (3, 17) = 1

gcd (4680, 15708) = 12.

We say a and b are relatively prime or coprime if gcd(a, b) = 1. For example, 3 and 17 are relatively prime. Note that gcd(a, b) = gcd(b, a). If a | b. then gcd(a, b) = a.

One of the ways to find gcd(a, b) is to compute the unique factorization of a and b. If we know the factorizations

$$a = P_1^{a_1} P_2^{a_2} .....P_r^{a_r} \quad \text{and}$$
$$b = P_1^{b_1} P_2^{b_2} .....P_r^{b_r}$$

then

$$d = \gcd(a, b) = p_1^{\min(a1, b1)} p_2^{\min(a2, b2)} \ldots\ldots p_r^{\min(ar, br)}$$

For example, let a = 4680 and b = 15708. Then

$$a = 4680 = 2^3 \cdot 3^2 \cdot 5 \cdot 13 = 2^3 \cdot 3^2 \cdot 5^1 \cdot \cdot 7^0 \cdot 11^0 \cdot 13^1 \cdot 17^0$$

$$b = 15708 = 2^2 \cdot 3 \cdot 7 \cdot 11 \cdot 17 = 2^2 \cdot 3^1 \cdot 5^0 \cdot 7^1 \cdot 11^1 \cdot 13^0 \cdot 17^1$$

so that

$$\gcd(4680, 15708) = 2^{\min(2, 3)} \cdot 3^{\min(2, 1)} \cdot 5^{\min(1, 0)} \cdot 7^{\min(0, 1)} \cdot 11^{\min(0, 1)} \cdot 13^{\min(1, 0)} \cdot 17^{\min(0, 1)}$$

$$= 2^2 \cdot 3^1 = 12.$$

For large numbers, finding factorization is not easy. For that we use Euclid's GCD Algorithm.

Given positive integers a > 0 and b, there exist unique quotient q and remainder r with

$$b = aq + r, \quad 0 \leq r < | b | .$$

Note that r = 0 if and only if b | a. Also q = r = 0 when b = 0 and we have 0 = a . 0 + 0. We use the following fact.

$$\gcd(a, b) = \gcd(a, b - aq) \qquad \text{for any } q$$

Now we define Euclid's GCD Algorithm and also illustrate with an example. Let b > a and a | b. Then

$b = aq + r,$	$0 < r < a$	$a = 4680, b = 15708$	
$a = rq_1 + r_1,$	$0 < r_1 < r$	$15708 = 3 \cdot 4680 + 1668,$	$0 < 1668 < 4680$
$r = r_1q_2 + r_2,$	$0 < r_2 < r_1$	$4680 = 2 \cdot 1668 + 1344,$	$0 < 1344 < 1668$
$r_1 = r_2q_3 + r_3,$	$0 < r_3 < r_2$	$1668 = 1 \cdot 1344 + 324,$	$0 < 324 < 1344$
$r_2 = r_3q_4 + r_4,$	$0 < r_4 < r_3$	$1344 = 4 \cdot 324 + 48,$	$0 < 48 < 324$
$r_3 = r_4q_5 + r_5,$	$0 < r_5 < r_4$	$324 = 6 \cdot 48 + 36,$	$0 < 36 < 48$
$r_4 = r_5q_6 + 0$		$48 = 1 \cdot 36 + 12,$	$0 < 12 < 36$
		$36 = 3 \cdot 12 + 0.$	

Here the last non-zero remainder, namely r_5 is the gcd (a, b). The novelty of this method is that we do not need to factor a and b.

We now introduce modular arithmetic. Let $m \geq 1$. We say a is congruent to b modulo m and write $a \equiv b \pmod{m}$ if $a - b$ is divisible by m or $a = b + km$ for some integer k. For example, $17 \equiv 2 \pmod{5}$ since $17 - 2 = 15$ is divisible by 5. It is easy to see that if a b (mod m), then b a (mod m). Basically a b (mod m) means both a and b has the same remainder when divided by b. One of the properties of modular arithmetic is the following:

If $a \equiv b \pmod{m}$ and $t \geq 1$, then $a^t \equiv b^t \pmod{m}$.

We can now state Euler's Theorem.

Theorem 1.2.1. Euler's Theorem: Let $m > 1$ be an integer and a be any integer coprime to m. Then

$$a^{\varphi(m)} \equiv 1 \pmod{m}$$

where $\varphi(m)$ is the Euler-totient function given by

$$\varphi(m) = m \prod_{\substack{p|M \\ p\ prime}} \left(1 - \frac{1}{p}\right)$$

For instance, let $m = 35$. Then we have $\varphi(m) = 35\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right) = 24$. Hence for any a coprime to 35, we have $a^{24} \equiv 1 \pmod{35}$. In particular 3^{24} has a remainder 1 when divided by 35. For our application in RSA, we will be considering m of the form m = pq where p, q are distinct primes. Then $\varphi(m) = \varphi(pq) = (p - 1)(q - 1)$. We state the RSA Cryptosystem now.

RSA Cryptosystem: Alice creates a public and private key as follow.

1. Choose two large prime numbers p and q and compute n = pq.
2. Keep p and q secret, known only to yourself, but make n public.
3. Choose an integer $1 < e < \varphi(n) = (p-1)(q-1)$ with the property that $(e, \varphi(n)) = 1$.
4. e is called the enciphering key.
5. The pair (n, e) is the Public key and is made known to everyone.
6. Compute the deciphering key d by solving the congruence
   $ed \equiv 1\{\mathrm{mod}\ \varphi(n)\}$ with $1 < d < (p - 1)(q - 1)$.
7. Deciphering key d must be kept private, known only to Alice

Sending a message to Alice:

1. Bob converts the message into a string of numbers M.
2. Bob uses public key (n, e) of Alice and compute $C \equiv M^e \pmod{n}$. C is the ciphertext.
3. The ciphertext C is transmitted to Alice.
4. Alice uses her private key d to get back the original message M by computing C^d(modulo n).

This works since $ed \equiv 1 \pmod{\varphi(n)}$ implies $ed = 1 + k\varphi(n)$ for some integer k and hence

$$C^d \equiv (M^e)^d = M^{ed} = M^{1+k\,\varphi(n)} = M^{(M\varphi(n))k} \equiv M \qquad \pmod{n}$$

by using Euler's Theorem.

We illustrate this with an example.

An Example:

1. Choose large primes $p = 71$ and $q = 101$. Then $n = 7171$ and $\varphi(n) = 70\ 100 = 7000$.

2. Choose an enciphering key $e = 37$; Check that $(37, 7000) = 1$.

3. Compute the deciphering key d by finding a solution to $37d \equiv 1$ (modulo 7000). The solution d with $1 < d < 7000$ is given by $d = 3973$ which is the deciphering key $d = 3973$.

3. (7171, 37) is the Public Key and 3973 is the private key.

Let message $M = 117$. The Ciphertext is

$$C = 117^{37} \equiv 227 \pmod{7171}.$$

This can be safely transmitted to me. Anyone who intercepts it will have to factor 7171 to decrypt it. Now use the decryption key by raising C to the 3973rd power and taking the result modulo 7171 to find

$$M = 227^{3973} \equiv 117 \pmod{7171}.$$

which is the original message M .

Suggested Project: Square and Multiply algorithm to compute $a^k \pmod{m}$ faster.

1.4 Binary Numbers

Binary numbers are base 2 numbers which are made up of only 0!s and 1!s. For example,

110100

is an example of a binary number. Like the usual numbers which are in base 10, the binary numbers are in base 2. Let us look for an example. Consider the number 123456. We have

$23456 = 2^5(1 + 732) = 2^5 + 2^5 \times 2^2(1 + 182) = 2^5 + 2^7 + 2^7 \times 2(1 + 91)$

$= 2^5 + 2^7 + 2^8 + 2^8(1 + 90) = 2^5 + 2^7 + 2^8 + 2^8 \times 2(1 + 44)$

$= 2^5 + 2^7 + 2^8 + 2^9 + 2^9 \times 2^2(1 + 10) = 2^5 + 2^7 + 2^8 + 2^9 + 2^{11} + 2^{11} \times 2(1 + 4)$

$= 2^5 + 2^7 + 2^8 + 2^9 + 2^{11} + 2^{12} + 2^{12} \times 2^2 = 2^5 + 2^7 + 2^8 + 2^9 + 2^{11} + 2^{12} + 2^{14}$

Writing 23456 as

$0 \times (2^0 + 2 + 2^2 + 2^3 + 2^4) + 2^5 + 0 \times 2^6 + 2^7 + 2^8 + 2^9 + 0 \times 2^{10} + 2^{11} + 2^{12} + 0 \times 2^{13} + 2^{14}$

we get the binary expansion of 23456 as

$23456 = (101101110100000)_2$

The digits are 0 and 1 and they are written starting with the coefficient of highest power of 2 on the right upto the coefficient of 2^0 on the right. The number of 0's on the right of binary expansion gives the exact power of 2 dividing the number. For example 5 is the exact power of 2 dividing 23456, we have five 0's on the right of the binary expansion of 23456.

Again, given a number in binary form as $N = (x_n x_{n-1} \,.....x_1 x_0)_2$, we get the decimal expansion of N by

$N = x_0 + x_1 2 + \cdots + x_{n-1}2^{n-1} + x_n{}^{2n}.$

For example, $N = (1010101010)_2$ in decimal notation is given by

$N = 0 + 1 \times 2^1 + 0 \times 2^2 + 1 \times 2^3 + 0 \times 2^4 + 1 \times 2^5 + 0 \times 2^6 + 1 \times 2^7 + 0 \times 2^8 + 1 \times 2^9$

$= 2^1 + 2^3 + 2^5 + 2^7 + 2^9 = 682.$

While writing in decimal notation, we can only sum the powers of 2 for which the corresponding coefficient is 1.

Exercises:

1. Write the following numbers in decimal notation.

   $(1010101100110)_2$, $(101011000110)_2$, $(101111100110)_2$, $(1000000000110)_2$

2. Write the following numbers in decimal notation.

654321, 1000001, 56237801, 2468097531, 963258741

3. Simplify the following and write in decimal notation.

$(1000101111100)_2 + (1100101000100)_2 \; (11101100100)_2$

4. Simplify the following and write in binary notation.

$(1111000110000)_2 \times 5642371$

1.5 Complex Numbers (Preliminary Idea only)

Complex numbers arise from trying to find square roots of real numbers. It is clear that the equation $x^2 + 1 = 0$ has no solution in real numbers. In other other words, -1 does not have a real square root. To overcome this problem, the concept of real and imaginary components of a numbers are introduced.

Let us denote by i a square root of 1 so that $i^2 = 1$. Then $(i)^2 = i^2 = 1$. The number i, called iota, has the property that

$$i^{4n} = 1, \qquad i^{4n+1} = i, \qquad i^{4n+2} = -1, \qquad i^{4n+3} = -i \qquad \text{for all } n \geq 0.$$

We define the set of Complex Numbers C as follow.

$$C = \{z = a + bi : a, b \in R\}$$

For $z = a + bi \in C$, we say a is the real part of z, denoted by Re(z), and b is the imaginary part of z, denoted by Im(z). For example, $1 + i$ is a Complex number with both real and imaginary parts equal to 1. Writing every real number $r \in R$ as $r = r + oi$, we see that the set of Real numbers is a subset of the set of Complex numbers.

We can view $z = a + bi$ as a polynomial $a + bx$ computed at $x = i$. Using the properties of powers of i, given two complex numbers $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$, we can define the sum $z_1 + z_2$ and product z_1z_2 as

$$z_1 + z_2 = (a_1 + b_1i) + (a_2 + b_2i) = (a_1 + a_2) + (b_1 + b_2)i$$

and

$$z_1z_2 = (a_1 + b_1i)\,(a_2 + b_2i) = a_1a_2 + a_1b_2i + b_1a_2i + b_1b_2i^2$$

$= a_1a_2 + (a_1b_2 + a_2b_1)i - b_1b_2 = (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i$

For example, $(1 + i) + (2 + 3i) = 3 + 4i$ and $(1 + i)(2 + 3i) = 2 + 3i + 2i + 3i^2 = (2 - 3) + (3 + 2)i = -1 + 5i$

Given a complex number $z = a + bi$, we define the complex conjugate $\bar{z} = a - bi$. This is the complex number whose imaginary part is negative of the imaginary part of z. For example, $2 - 5i = 2 + 5i$. For $\bar{z} = r \in R$, we have $\bar{z} = z = r$ as the $Im(z) = 0$ in that case.

For complex numbers z_1 and z_2, we have

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \qquad \text{and} \qquad \overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$

Also for $z = a + bi$, we have

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2 \geq 0$$

since a, b are reals. Define the absolute value or modulus of $z = a + bi$, denoted by $|z|$, as

$$\sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$$

where we take the non-negative square root $a^2 + b^2$. For example, $|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$.

Following are some properties of the absolute value of complex numbers. Here z, z_1, z_2 are complex numbers.

1. $|z| = |-z| = |\bar{z}| \geq 0$ for all $z \in C$.
2. $|z| = 0$ if and only if $z = 0 = 0 + 0i$.
3. $|z_1z_2| = |z_1|\,|z_2|$ for all $z_1, z_2 \in C$.
4. Triangle inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$ for all $z_1, z_2 \in C$.

   For $z = a + bi \neq 0$, we have

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{bi}{a^2 + b^2}$$

For example, $(2 + 3i)^{-1} = \frac{2}{13} - \frac{3i}{13}$. For $z_1 - 1, z_2 \in C$ with $z_2 \neq 0$, we have

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{z_1 \bar{z}_2}{|z_2|^2}$$

As an example, we have

$$\frac{2+3i}{4-5i} = \frac{(2+3i)(4+5i)}{4^2+5^2} = \frac{-7+22i}{41} = \frac{-7}{41} + \frac{22i}{41}$$

Given $z \in C$, $z \neq 0$, we have

$$\frac{z}{|z|} = \frac{a+bi}{a^2+b^2} = \frac{a}{a^2+b^2} + \frac{bi}{a^2+b^2}$$

Let $0 \leq \theta < 2\pi$ be such that $\sin\theta = \frac{b}{a^2+b^2}$ and $\cos\theta = \frac{a}{a^2+b^2}$. The angle θ is called the argument of z and we have

$$z = |z|(\cos\theta + \sin\theta\, i) = |z| e^{\theta i}$$

which is called the Euler formula for the complex number z. For example,

$$1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = |z|\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4} i\right) = \sqrt{2} e^{\frac{\pi}{4} i}$$

Exercises:

1. Find the complex conjugates and modulus of the following complex numbers.

   $1 - i$, $10 + 4i$, $(3 + 5i)(4 + 6i)$, $\frac{2+7i}{5+4i}$

2. Compute z, z^2, z^3, z^{-1} for the following z:

   $1-i$, $3 + i$, $4 + 6i$, $\frac{9+2i}{2+9i}$, $1 + \pi i$, $\sqrt{3} + \sqrt{6i}$

1.6 Indices, Logarithm and Antilogarithm

Indices

A power of a number is the product of a certain number of factors, all of which are the same. For example, 5^9 is a power, in which the number 5 is called the base and the number 9 is called the index or exponent. In fact, for any a, we have

$a^1 = a$

$a^2 = a \cdot a$

$a^3 = a \cdot a \cdot a$

.....

$a^n = a \cdot a \cdot a \cdot a \cdot a.....$ n times

for any n. We have $2^4 = 2.2.2.2 = 16$ for example.

Let a and b be real numbers and m and n be integers. The Indices satisfy the following rules:

1. A Zero power is given by $a^0 = 1$ for $a \neq 0$. For example $4^0 = 1$. We note that 0^0 is not defined. It is sometimes called an indeterminant form.

2. For a positive n, the negative power a^{-n} is defined by

$$a^{-n} = \frac{1}{a^n}$$

For example, $a^{-3} = \frac{1}{a^3}$. In particular $2^{-1} = \frac{1}{2}$, $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$, $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ and so on.

3. A Fractional power, denoted by $a^{\frac{1}{n}} = \sqrt[n]{a}$, is given by

$$\left(\sqrt[n]{a}\right)^n = a$$

For example, $9^{\frac{1}{2}} = \sqrt{9} = 3$ and $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

All indices satisfy the following laws. Let a and b be real numbers and m and n be rational numbers.

1. To multiply powers with the same base, add the indices.

$$a^m a^n = a^{m+n}$$

For example, $2^3 . 2^2 = 2^5 = 32$

2. To divide powers with the same base, subtract the indices.

$$\frac{a^m}{a^n} = a^{m-n}$$

For Example, $\frac{2^3}{2^2} = 2^{3-2}$ = 2 and $\frac{2^2}{2^3} = 2^{2-3} = 2^{-1} = \frac{1}{2}$. Note $\frac{a^m}{a^m} = a^{m-m}$ = a^0 = 1 for $a \neq 0$

3. To raise a power to a power, multiply the indices.

$$(a^m)^n = a^{mn}$$

For example, $(2^2)^3 = 2^6 = 64$

4. A power of a product is the product of the powers.

$$(ab)^m = a^m b^m$$

For example, $3^2 . 4^2 = (3 . 4)^2 = 12^2 = 144$.

5. A power of a quotient is the quotient of the powers.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad \text{when } b \neq 0$$

For example, $\frac{6^2}{2^2} = \left(\frac{6}{2}\right)^2 = 3^2 = 9$

Simplify the following by the above rules.

1. $a = x^{\frac{1}{5}}.x^{\frac{4}{5}}$

2. $a = x^2 \div x^{\frac{3}{2}}$

3. $a = \left(x^{\frac{5}{3}}\right)^6$

4. $a = \frac{x^3 y^4}{x^5 y^2}$

1.7 Logarithms

The Logarithm is the inverse image of an index. The logarithm of any positive number to a given base (a positive number not equal to 1) is the index of the power of base which is equal to that number. If N and b $\neq$ 1 are any two positive real numbers and for some real x, $b^x = N$, then x is called the logarithm of N to the base b. It is written as $\log_b N = x$. That is, if $N = b^x$, then $\log_b N = x$. Since $3^4 = 81$, the value of $\log_3 81 = 4$. Some examples:

1. $\log_{10} 0.01 = -2$ since $10^{-2} = 0.01$

2. $\log_2 \sqrt{2} = \frac{1}{2}$ since $\sqrt{2} = 2^{\frac{1}{2}}$

3. $\log_b b = 1$ since $b^1 = b$ for any $b > 0, b \neq 1$.

4. $\log_b 1 = 0$ since $b^0 = 1$ for any $b > 0, b \neq 1$.

From the definition of logarithms(logs), we obtain the following for $a > 0, b > 0, b \neq 1$.

$$\log_b b^n = n \quad \text{and} \quad b^{\log_b a} = a$$

System of logarithms: There are two systems of logarithms, natural logarithm and common logarithms which are used most often.

1. Natural Logarithm: These were discovered by Napier. They are calculated with respect to the base e which is approximately equal to 2.718. We usually denote $\log_e x$ by ln x

2. Common Logarithms: Logarithms to the base 10 are known as common logarithms.

Here we list some facts about logarithms:

1. Logs are defined only for positive real numbers.

2. Logs are defined only for positive bases different from 1.

3. In $\log_b a$, neither a nor b is negative but the value of $\log_b a$ can be negative. For example, $\log_{10}0.01 = -2$ since $10^{-2} = 0.01$.

4. Logs of different numbers to the same base are different, i.e. if $a \neq c$, then $\log_b a \neq \log_b c$.

   In other words, if $\log_b a = \log_b c$, then $a = c$.

5. Logs of the same number to different bases have different values i.e. if $a \neq b$, then $\log_a c \neq \log_b c$. In other words, if $\log_a c = \log_b c$, then $a = b$.

Some important properties of logarithms:

1. Logarithm of a Product:

$$\log_b (MN) = \log_b M + \log_b N$$

   This follows from the property $b^{m+n} = b^m b^n$. As an example, we have $\log_2 (4 \cdot 8) = \log_2 4 + \log_2 8 = 2 + 3 = 5$. If the product has many factors, we just add the individual logarithms:

$$\log_b(ABCD) = \log_b A + \log_b B + \log_b C + \log_b D$$

2. In particular, we get the Logarithm of a power:

$$\log_b(a^n) = n \log_b a$$

   Hence for example $\log_2(3^{100}) = 100 \log 2_3$

3. Logarithm of a quotient:

$$\log_b \left(\frac{M}{N}\right) = \log_b M - \log_b N$$

   This also follows from the property $b^{m-n} = \frac{b^m}{b^n}$. For example

$$\log_3 \left(\frac{81}{8}\right) = \log_3 81 - \log_3 2^3 = 4 - 3 \log_3 2$$

4. Logarithm in two different bases b_1 and b_2:

$$\log_{b2} N = (\log_{b1} N)(\log_{b2} b_1)$$

   In particular, when $b_1 = e$ and $b_2 = 10$, we have

$\log_{10} N = (\log_e N)(\log_{10} e) = 0.434 \log_e N$ and $\log_e N = (\log_{10} N)(\log_e 10)$

$= 2.303 \log_{10} N$

Exercises:

1. Expand $\log_b \left(\dfrac{a^a b^b}{c^c d^d}\right)$

2. Expand $\log_b \left(\dfrac{4x^6}{9y^7}\right)$

3. Simplify $\log_{10} a + \log_{10} b^2 + \log_{10} c^3$

4. Simplify $\log_a a - \log_b b^2 + \log_c c^3 - \log_d d^4$

Anti-logarithm

The anti-logarithm of a number is the inverse process of finding the logarithms of the same number. If x is the logarithm of a number y with a given base b, then y is the anti-logarithm of (antilog) of x to the base b.

If $\log_b y = x$, then y = antilog of x.

Natural Logarithms and Anti-Logarithms have their base as 2.7183. The Logarithms and Anti-Logarithms with base 10 can be converted into natural Logarithms and Anti-Logarithms by multiplying it by 2.303.

The Zero Index

We have $\dfrac{9^7}{9^7} = 1$. On the other hand, applying the above index law 2 and ignoring the condition $m > n$, we have $\dfrac{9^7}{9^7} = 9^0$. If the index laws are to be applied in this situation, then we need to define 9^0 to be 1. More generally, if $a \neq 0$, then we define $a^0 = 1$. We note that 0^0 is not defined. It is sometimes called an indeterminant form.

The index laws are also valid for the zero index. And for any non-zero a and b, we have

$$(7a^3b^2)^0 = 1$$

Negative Exponents

Let's look at the decreasing powers of 2. We have

$$2^5 = 3^2,\ 2^4 = 16,\ 2^3 = 8,\ 2^2 = 4,\ 2^1 = 2,\ 2^0 = 1,\ 2^{-1} = ?,\ 2^{-2} = ?$$

As we can see, at every step when we decrease the index, the number is halved. Therefore it makes sense to define

$$2^{-1} = \frac{1}{2}$$

Further, continuing the pattern, we define

$$2^{-2} = \frac{1}{4} = \frac{1}{2^2},\ 2^{-3} = \frac{1}{8} = \frac{1}{2^3},\quad \text{and soon}$$

1.8 Laws and Properties of Logarithms

Logarithms are useful in may domains, particularly in solving exponential equations. For example, we use logarithms to measure Richter scale in earthquakes, decibel measures in sound, pH balance in Chemistry and the brightness of stars, to name a few.

Let us look at the example of how logarithms are in used in measuring the magnitude of earthquakes. The energy released by an earthquake gives the magnitude of the earthquake.

The Richter magnitude scale (commonly known as Richter scale) is used to measure this magnitude of an earthquake. Seismographs detect movement in the earth's surface and measures the amplitute of the earthquake wave. Let λ be the measure of the earthquake wave amplitute and λ_0 be the measure of smallest detectable wave (or the standard wave).
Then the Richter Scale is given by the formula

$$R = \log_{10}\left(\frac{\lambda}{\lambda_0}\right)$$

Higher the Richter scale, more is the intensity of the earthquake and damages caused. Usually earthquakes of Richter scale up to 4.9 does not cause damage. The earthquakes of Richter scale 6-6.9 and above cause major damages. The strongest earthquake till date was recorded in Chile in 1960 with Richter Scale of 9.5 which caused severe damage.

Example 1: There was an earthquake with a wave amplitude 2020 times the wave. Calculate the Richter scale with two decimal digits?

Solution: We have $\lambda = 2020\lambda_0$ This gives

$$R = \log_{10}(\frac{\lambda}{\lambda_0}) = \log_{10} 2020 = 3.3.$$

Hence the Richter scale of the earthquake is 3:3.

Example 2: A scientist running an experiment finds that a particular bacterial colony doubles its population every 20 hours. He starts with 200 bacteria cells. She expects the number of cells to be given by the formula $b = 200(\sqrt{2})^{\frac{t}{20}}$ where t is the number of hours for which the experiment is running. Find the number of hours after which there will be 500 bacteria cells.

Solution: Taking logarithms on both sides of $b = 200(\sqrt{2})^{\frac{t}{20}}$ and putting b = 500, we get

$$\log 500 = \log b = \log 200 + \log(\sqrt{2})^{\frac{t}{20}} = \log 200 + \frac{t}{20}\log\sqrt{2} = \log 200 + \frac{t}{40}\log 2.$$

This gives

$$t = 40 \times \frac{\log 500 - \log 200}{\log 2} = \frac{40\log\frac{500}{200}}{\log 2} = \log\frac{40\log\frac{5}{2}}{\log 2} = 22.964.$$

Therefore after 23 hours, there will be 500 bacteria cells.

Exercises: Let the population of the world in t years after 2010 be given by the formula $P = 4.7(1.02)^t$ billions.

i) Calculate the total population of the world in the year 2029 to the nearest million.

ii) Find the year in which the population will be double of the population of 2020.

Chapter II

Numerical Applications

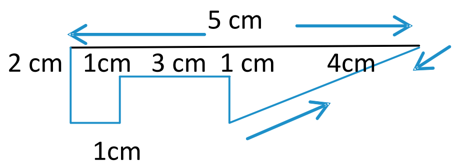
Learning Objectives:

After completion of this unit, Students will be able to:

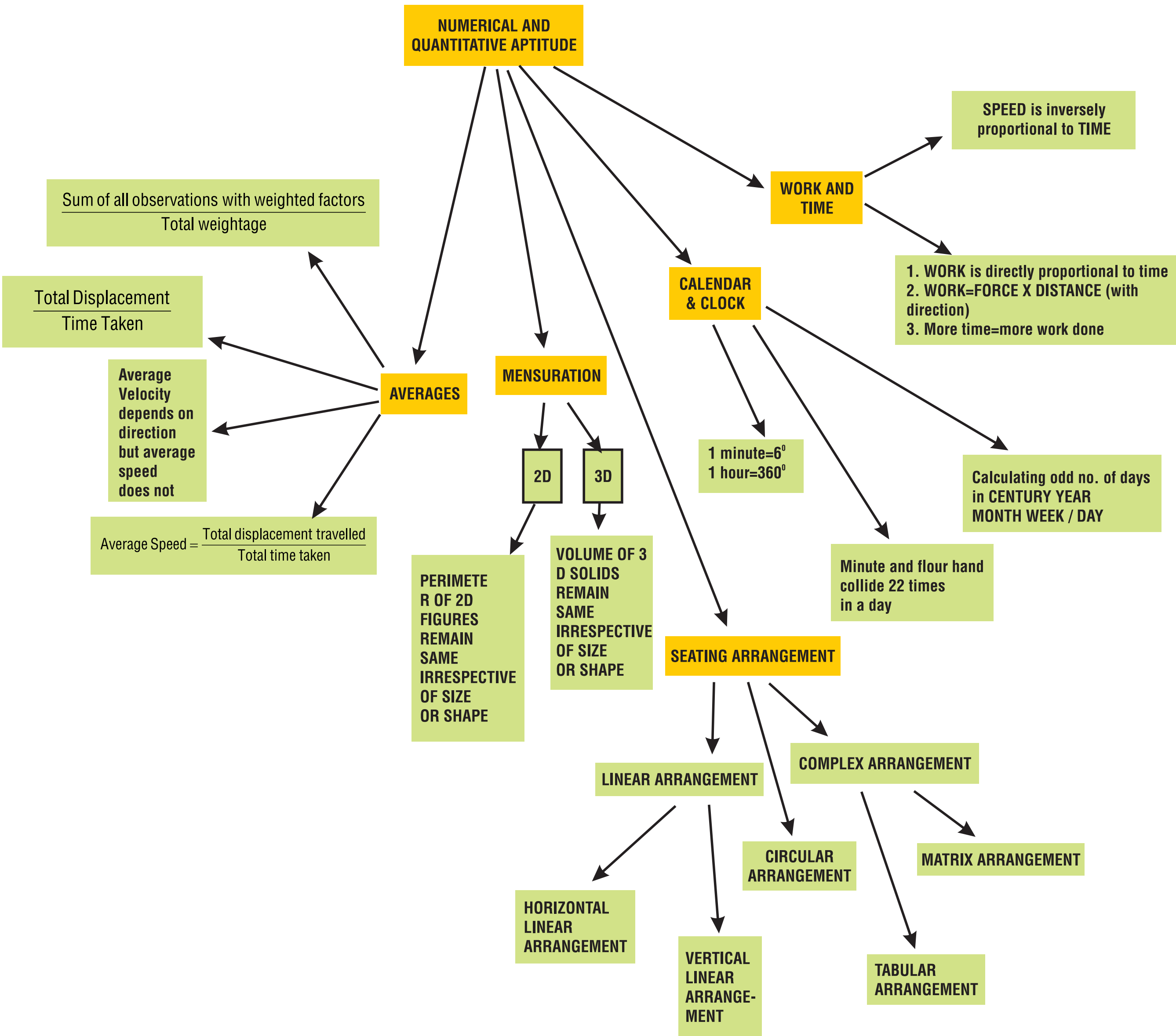
- Differentiate between average speed and average velocity in order to analyse the role of direction in speed.
- Establish the relationship between work and time in order to relate the efficiency and factors related to it.
- Determine the odd number of days in a month/year/century so as to decode it and find out the day of week for the same.
- Evaluate the angular value of a minute in order to compare the speed of the minute hand and hour hand.
- Derive the formulas for surface area and volume of combined solids in order to relate the same with real world problems.
- Compare the positive work done and negative work done in order to find workable solution for the posed problem.

Before You Start:

	You should know how to :	Juggle Your Brain
1.	Calculate mean (average) of given data Mean = $\dfrac{\sum_{i=1}^{n} fixi}{\sum_{i=1}^{n} fi}$	Find the mean marks if marks scored by 7 students are 38, 72, 91, 76, 25, 99, 65 respectively (out of 100)
2.	Determine the distance covered and the displacement of the body	An Artificial satellite moves in a circular orbit of radius 172 km and reaches to its initial position in 190 days. Find out the total distance travelled by the body in 1 revolution.
3.	Classify the given year as a leap year or a non-leap year.	Year 2002 was a __________ year.
4.	a) What is the measure of angle formed by clock hands in an hour? b) Convert days/hours/minutes/seconds accordingly.	a) In a clock minute hand points on 5 and hour hand points on 3, the region enclosed between them is known as _______. b) $7\frac{3}{8}$ Hours = 7 Hours __Minutes ___seconds.
5.	a) Differentiate between 2D and 3D figures	Given situation a) A right Δ with sides 3 cm, 4 cm and 5 cm is revolved about the side 4 cm.

	b) Find Perimeter/Area of combined 2D figures	I) Name the solid figure obtained. ii) Categorize it as 2D or 3D iii) Now If Δ was revolved about the side 5 cm. Draw figure and how it is different from the part (I) b) Find the perimeter of the given park. 
6.	Determine Speed $\text{Speed} = \dfrac{\text{Distance travelled}}{\text{Time taken}}$	A train travels 785 km in 3 hours 45 minutes find the Speed of the train. (Assuming that the train is moving at uniform speed).

Concept Map:



2.1 Introduction

Numerical and quantitative aptitude is a core concept that helps to deal with our day to day problems and find out workable solutions to those problems.

Numerical ability relates to number and its various operations. Quantitative Aptitude comprises of mathematical operations related to logic. It is a way or a language to express the logic along with the numerical approach. It is a skill rather than the concept which enhances analytical thinking to devise simple and more ways to find the solutions to the real-world situations. Also, it focuses on enhancing the problem-solving skill by finding simple and short approach through computation ability.

Numerical Skills:

CONTENT:

A Average

B Clock and Calendar

C Work, Time and Distance

D Mensuration

E Seating Arrangements

2.2 Average

Introductory Discussion:

Four football matches are played between two teams, team A and team B and their scores are:

SCORES TEAM A	10	5	4	6
SCORES TEAM B	5	2	8	5

a) What is the final score of team A and team B?

b) Which team scores better? Give reason.

$$Average\ (\text{Mean}) = \frac{Sum\ of\ all\ observations}{Total\ number\ of\ observations}$$

'Average or Mean' is a statistical measure, often known as a measure of central tendency of given data. It represents a calculated central value of the given data Average is also known as arithmetic mean or mean.

Mean is denoted by $\bar{x}$

More about Average:

1. Average lies between maximum and minimum value of given set of observations.

2. If value of each observation is increased or decreased by the same value N then the average will also be increased or decreased by the same value.

3. If value of each observation is multiplied or divided by the same value N then the mean will also be multiplied or divided by the same value.

Weighted Average

Example 1

The average monthly savings of a company was Rs. 12 Lakh for the first 3 months, Rs. 12.5 Lakh for the next 4 months and Rs. 31.2 Lakh during the next 5 months of a year. Total expenditure during the year is 78 Lakh. Find the average monthly earnings of the company.

Total Annual savings = 12 Lakh x 3 + 12.5 Lakh x 4 + 31.2 Lakh x 5

= (36 + 50 + 156) Lakh

= 242 Lakh

Total yearly expenditure = 78 Lakh

Total yearly income = 242 + 78 = 320 Lakh

Average monthly income of company = $\frac{320}{12} lakh = 26.67$ Lakh (approximately)

In the above example we observe how the concept of average plays a vital role in the planning of the family budget to the planning of investment in the company. So, the need arises as to what percentage of the capital amount is to be invested in salaries, infrastructure etc. and on an average what amount is to be spent? So answer to these questions is weighted mean/average.

Weighted mean (average) is similar to arithmetic mean ($\bar{x}$), except that instead of all elements having equal weight in the case of arithmetic mean, some elements with high weight contribute more than the elements with low weight.

Weighted mean ($\bar{x}_w$) is the mean of a data set of 'n' elements when some elements carry more weight than other

$$\frac{\sum_{i=1}^{n} w_i . x_i}{\sum_{i=1}^{n} w_i}$$

where

n= no of elements in data set

x_i = value of i^{th} element

w_i = weight of i^{th} element

$\bar{x}_w$= weighted mean

Note : If all the weights are equal then weighted mean = arithmetic mean

Example :2

A school follows the following criterion for grading a student for annual result:

ASSESSMENT	WEIGHTAGE
Homework	25%
Quiz	30%
Test	10%
Final exams	35%
	100%

In his math submissions, Joy scored 88 marks in homework, 71 marks in class quiz ,97 marks in tests conducted throughout the session and 90 marks in the final exams. What is Joy's annual result in Mathematics?

Weighted mean = $\bar{x}_w = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$

$= \frac{25 \times 88 + 30 \times 71 + 10 \times 97 + 35 \times 90}{100}$

$= \frac{8450}{100}$ $\boxed{\bar{x}_w = 84.5\ \%}$

Example : 3

The average of 4 positive integers is 39. The highest integer is 73 and the lowest integer is 19. The difference between the remaining two integers in 18. Find the remaining integers.

AVERAGE (MEAN) = $\frac{\textit{Sum of all observations}}{\textit{total number of observations}}$

Let remaining two integers be x, x+18

$$39 = \frac{73 + X + X + 18 + 19}{4}$$

46 = 2x

X= 23
X+18 = 41

Hence, remaining integers are 23 and 41.

Example: 4

Five years ago, average age of a family of five was 32 years. The family includes Mrs. and Mr. Rai, their son, daughter and daughter in law. Recently, The daughter, Priya moved out of the house after marriage, at the age of 30 years. Calculate the family's present average age.

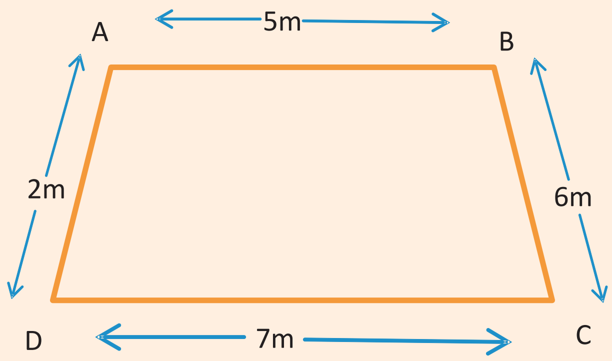
Total age of 5 family members 5 years ago	=	5 x 32 = 160 years
Total age of 5 family members today	=	160 + 25 = 185 years
Total age of 4 family members as Priya left	=	185 – 30 = 155 years
Present average age of Family	=	$\frac{155}{4}$ = 38.75 years

Differentiate Between Average Speed And Average Velocity:

AVERAGE SPEED	AVERAGE VELOCITY
Average Speed = $\frac{\text{Total distance travelled}}{\text{Total time taken}}$	Average Velocity = $\frac{\text{Total displacement of body}}{\text{Total time taken}}$

Example: 5

An insect starts from a point A and covers distance AB in 4 seconds. Then covers distance BC in 7 seconds, distance CD in 5seconds and distance DA in 2 seconds. Calculate the average velocity and average Speed.



Total Time Taken = 18 seconds

Total Distance Travelled = 20 m

Total Displacement = 0 m (as the final and initial position is same)

$$\text{Average Speed} = \frac{\text{Total distance travelled}}{\text{Total time taken}} = \frac{20}{18} = 1.11 \text{ m/s (approximately)}$$

$$\text{Average velocity} = \frac{\text{Total displacement}}{\text{Total time taken}} = \frac{0}{18} = 0 \text{ m/s}$$

Exercise: 1A

1. There are 44 boys and 36 girls in a class. The Average marks of boys are 40 and that of girls is 38. Find the average marks of the class?
2. The average of five numbers is 87. If one of the numbers is excluded, then the average gets decreased by 5. Find the excluded number.
3. A nursery is closed on Sunday. The average plants sold in the remaining six days of a week is 156 plants and the average sale from Monday to Friday is 124 plants. Find the number of plants sold on Saturday?
4. The average of five consecutive numbers is 125. Find the product of the number at the extreme positions.

5. Under MNREGA Schema 1000 new labourers are enrolled in Delhi. Earlier they were getting Rs 200 as daily wages, but now the authorities have increased the budget for them by 15 Lakh per month.

   a) Calculate the present monthly budget of the ministry for 1000 laborer

   b) Find the increase in daily income due to budget increase

   c) Find the new Average monthly income per labour.

6. The mean of 35 observations was found to be 98.6. But later, it was found that 72 was misread as 27. Find the correct mean.

7. The average annual PF contribution of certain number of defence officers is Rs28560 and that of other officers is Rs 22500. The number of defence officers is 22 times that of other officers. Then find the average savings of all the officers in total.

8. 5 years ago, the average age of the 3 children of Mr. Pandey was 8 years. A new baby is born in the family now. Find the present average age of the family.

9. In a restaurant 35 visitors can have lunch at a time. If the number of visitors increases by 7,then the expense of the restaurant on food increases by Rs. 42, while the average expenditure per head decreases by Rs. 1. Find the original expenditure of the restaurant.

10. The average of runs of Virat Kohli, the famous cricket player, of last 10 innings were 72. How many runs he must make in 11th inning to increase the average by 3 runs.

11. An ant is moving around a circular path of radius 3.5 cm and takes 3 seconds to complete 1 revolution. Find the average speed and average velocity?

12. Sara walks 7.2 km in one and half hour and 3.5 km in 2 hours in the same direction.

    What is the Sara's average speed for the whole journey?

2.3 Clock And Calendar

Introductory Discussion

Lalita Babar is three times winner of Mumbai Marathon. She won Bronze medal in 3000 m steeplechase at the 2014 Asian Games and broke the National record with the time of 9 hours 35 minutes and 37 seconds.

Time To Work:

1. Write the length of steeple chase race in decimeters.

2. Convert the time taken in seconds.

For thousands of years people measured time by observing the passage of day and night, the stars and the change of seasons. But it is very important to measure the time accurately. This lead to the inventions of tools for measuring time such as Sundial, hourglass, water clock etc. Many devices were invented over the centuries to measure the time accurately, and finally in due course of time Clock was invented. The most accurate clock in the world is The Cesium Fountain Atomic Clock developed at NIST Laboratories in Colorado, USA.

Units of Time :

Time is measured in seconds, minutes, hours, days and so on. Question arises as to how these units were derived.

The units of time we use are based on the sun, the moon and the earth rotation and revolution.

Common Units Related To Time :

1 Minute = 60 seconds	
1 hour = 60 minutes = 3600 seconds	
1 day = 24 hours =1440 min =86400 seconds	
1 week =7 days	
1 year = 12 months = $365\frac{1}{4}$ *days*	
1 Decade = 10 years	
1 Century = 100 years	
1 Millennium = 1000 years	

Also,

1 millisecond = $\frac{1}{1000}$ = 10^{-3} second

1 microsecond = $\frac{1}{1000000}$ second = 10^{-6} second

Example : 6

What would be the time four and a half hour before 2.15 pm?

= 2:15 pm - $4\frac{1}{2}$ hours

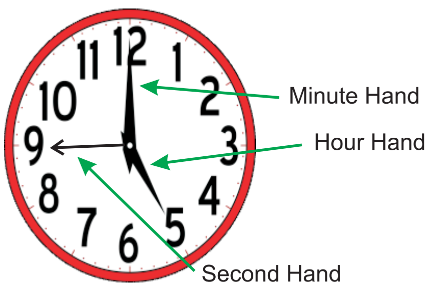
= 2:15 pm - (4 hours + 30 minutes)

= 10:15 am - 30 minutes

= 9:45 am

Structure of Clock

A clock is device provided with three hands as an hour hand, minute hand and seconds hand.



"HOROLOGY"
Study and measurement of time and making of clocks

A clock hand makes an angle of 360^0 in one revolution

Angle of clock hand in one revolution = 360^0

Therefore,

Angular value of an hour hand in one hour = $\frac{360^0}{12} = 30^0$

$\Rightarrow$ $\boxed{\text{Speed of Hour hand} = 30^0 \text{ per hour}}$

Angular value of minute hand in one minute $= \frac{360^0}{60} = 6^0$

$\Rightarrow$ $\boxed{\text{Speed of minute hand} = 6^0 \text{ per minute}}$

Angular value of a second hand in one second $= \frac{360^0}{60} = 6^0$

$\Rightarrow$ $\boxed{\text{Speed of second hand} = 6^0 \text{ per second}}$

Collisions of Hands of A Clock

Two hands of clock collide when angle between them is 0°

Let's find how many times in a day the minute and hour hands of a clock coincide with each other?

But how?

Number of hours in a day = 24

So, hour hand completes two revolutions in a day

first collision is at : 12:00 am

Second collision is at 1: 05 am

Therefore, minute and hour hands must collide 24 times a day but Hands will not collide between 11'o Clock to 12'o clock. This will happen 2 times a day. Hence, 24 - 2 = 22 times. Answer: 22 times a day.

Time of Collision

As Hour and Minute hands collide 22 times in 24 hours.

Time of one collusion = $\frac{24}{12}\ hour$

Therefore, Time of one collision = $\frac{12}{11}\ hour$

$= \frac{12 \times 60}{11}$ minutes

$= \frac{720}{11}$ minutes

$= 65\frac{5}{11}$ = 65.45 minutes (approx.)

So, if 1st collision is at 12'O clock, then next collision will be after $65\frac{5}{11}$ minutes after 12'o clock

Stretch Your Brain:

Complete the following table on the basis of the above information:

FREQUENCY	Time in fractions	Exact time	Time that Appears on clock (approx.)
1st	12:00:00	12:00:00	12:00
2nd	$1:05:\frac{5}{11}$	1:05:27	1:05
3rd	$2:10:\frac{10}{11}$	2:10:54	2:10
4th			
5th	$4:21:\frac{9}{11}$	4:21:16	4:20
6th			
7th			
8th			
9th			
10th			
11th	$11:59:\frac{11}{11}$	12:00:00	12:00:00

Investigation 1: Calculate the speed of seconds hand?

Investigation 2: Find out the difference in the speed of minute hand and hour hand ?

Relation Between Time and Angle Between the Hands of Clock:

Angle in 12 hours = 360^0

Angle in 1 hour = 30^0

Angle in 1 minute = $\frac{30^0}{60^0} = \left(\frac{1}{2}\right)^0$

Hour hand = $\left(30H + \frac{m}{2}\right)^0$

But 1 minute = 6^0

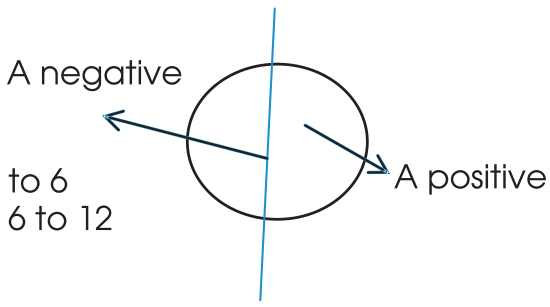
So, Angle between hands of clock(A) = $30H + \frac{m}{2} - 6m$

H means hour shown by hour hand

m means minutes shown by minute hand.

$T = \frac{2}{11}(H \times 30^\circ \pm A)$ or $A = 30^\circ \times H - \frac{11}{2} \times t$ (t = time in minutes indicated by minute hand)

T = time at which angle is formed
H = running hour
A = angle between hands
A is positiveif hands collide between 12 to 6
And negative if hands collide between 6 to 12



Note: Value of A is always taken as positive, negative sign is ignored.

Example 7

At what time between 4'o clock and 5'o clock, will the hands of a clock make an angle of 20°

$H = 4 \qquad A = 20^\circ$

$T = \frac{2}{11}[4 \times 30^\circ \pm A]$

$= \frac{2}{11}[4 \times 30^\circ + 20^\circ]$

A is (positive because the hands of clock lies in between 12 to 6 first half of clock)

$= \frac{2}{11}[120^\circ + 20^\circ]$

$= \frac{2}{11}[140^0]$

$= \frac{280^0}{11} = 25\frac{5}{11}$ = 25 minutes $\frac{5}{11}$ seconds

= 25 minutes 27 seconds

Time = 4:25:27

Example 8

At what time between 3'o clock and 4'o clock will the hands of a clock collide?

At 3'o clock hour hand is at 3 and minute hand is at 12. i.e. the minute hand is 15 min. space apart from hour hand.

Hence the hand will collide, if minute hand covers 15 minutes over the hour hand.

As, Time of one collision = $\frac{12}{11}$ hours

Therefore 15 minutes will be covered in $\left(\frac{12}{11} \times 15\right) = \frac{180}{11} = 16\frac{4}{11}$ minutes

So, the hands will collide at $16\frac{4}{11}$ minutes past 3.

Example 9

Find the angle between the hands of clock when it struck 15 minutes past 7?

$A = 30° \times H - \frac{11}{2} T$

$= 30^0 \times 7 - \frac{11}{2} \times 15$

$= 210^0 - 82.5^0$

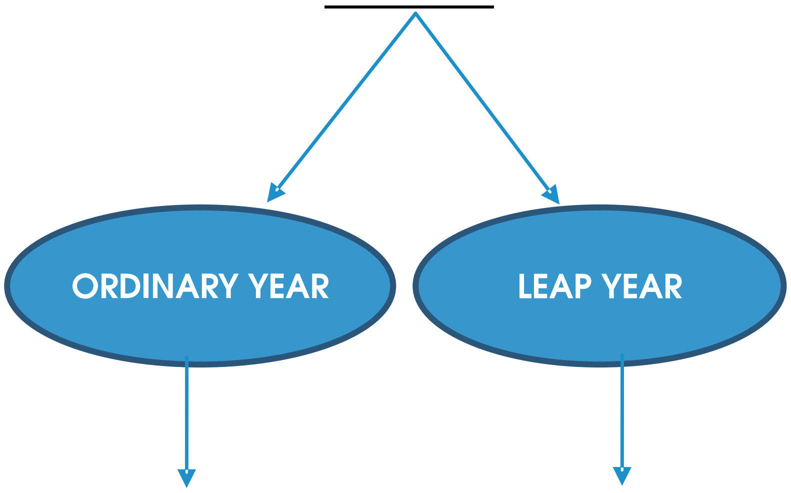
$= 127.5^0$

Calendar

Clock apprises us about the time span for seconds, minutes, day etc. whereas calendar depicts the days, weeks and months of a specific year.

Structure of Calendar

Calendar



No. of Days	365	366
No. of weeks	52 Weeks and 1 Day	52 weeks and 2 Days
No. of odd days	1	2

ODD DAY* The extra day(s) left after exact number of weeks is considered as an Odd day.

Calculating Number of Odd Days in a Month
Juggle Your Brain

MONTH	NO. OF ODD DAYS
JAN	3
FEB	0/1
MARCH	3
APR	2
MAY	
JUNE	
JULY	
AUG	
SEP	
OCT	
NOV	
DEC	

January has 31 days = 4 weeks + 3 days
January has 3 odd days

Complete the table
$\frac{30}{7}$ = 4 weeks and 2 days

February has either 28 or 29 days =4 weeks + 0 day or 4 weeks + 1 day
Februray has eighter 0 or 1 odd days

Decoding for Day of Week

Coding and Decoding is one of most important parameters of day to day life. As nowadays most of the thing we use is based on cryptogamy (coding of data) Example our mobile phones passwords, debit cards, computers etc. uses this coding methodology and so the need arises for decoding of the same.

Decoding the day when date is given is done on the basis of odd number. of days in the month/year/century. If we consider that the week begins with a Monday and ends on a Sunday, then decoding is done as :

Code	Day
0	Sunday
1	Monday
2	Tuesday
3	Wednesday
4	Thursday
5	Friday
6	Saturday

Example :10

If 30^{th}March 2020 was Monday. What would be the day after 61 days?
Reference date : 30 March 2020
Reference day : Monday
Duration : 61 days
61 days : 8 weeks and 5 days

$\frac{61}{7}$ = 8 weeks and 5 days

So the day after the 8^{th} week would be the same i.e. 56^{TH}Day and it Will be MONDAY. 57^{th} day will be TUESDAY.
Therefore, the 61^{st} day will be a Saturday.

Leap Year:

A Leap year occurs every four years, To check whether the given year is a leap year or not, divide the year by 4 and if the remainder comes as 0, then the year is a leap year

FUN FACT
A number is divisible by 4 if the last two digits are divisible by 4

Calculating Number Of Odd Days In A Century :

ILLUSTRATION: What day of the week was on 31st DECEMBER 100 A.D.?

STEP 1	Write the given year as multiple of 400	Total number of years 100 Number of leap years in 100 years = 24 years because 100/4 = 25 but 100 is divisible by 4 so (25 – 1 = 24)
STEP 2	Find the no. of ordinary years and leap years in a century.	So, ❖ Total leap year = 25-1 = 24 ❖ Ordinary years = 100-24 =76
STEP 3	Write the total no. of odd days in a century	Number of Odd days in a century = 76x1 + 24x2 = 76+ 48 = 124 days
STEP 4	Dividing the total no. of odd days by7 remainder indicates the last day of the year.	= 124/7 = 17 weeks and 5 odd days = Dec 31^{st} = Friday

Number Of Odd Days In A Century

Complete the table

Stretch Your Mind:

Century	No. of Odd Day		Last Day of Century
100	5		Friday
200	5+5=10=7x1+3=	3	Wednesday
300	5x3=15=7x2+1=	1	Monday
400	5x4=20 multiple of 4	0	Sunday
500		5	Friday
600			
700		1	
800		0	
1000	5+5=10=7x1+3	3	
1400			
1600			
1700			
2000			
2100			

Exercise 1(b)

1. If today is a Tuesday, what will be the day on 7706th day?
2. Find the total number of days from 26th January 2008 to 15 May 2008?
3. If the second day of April month is a Friday, then find the last day of the next month?
4. Workout for the day of week on the given date:

   (1) 15th August 1947

   (2) 22nd November 2025

   (3) 21st September 2080

   (4) 18thOctober 2100

5. A local train from Mumbai leaves every 40 minutes from the station. When inquired by a passenger, the help desk executive informed that the train had already left 10 minutes ago. If this information was given at 10:15 a.m.

   a) At what time did the train leave the station?

   b) At what time will the next train leave the station?

   c) For how long will that man has to wait for the next train?

   d) Find the angle formed between the minute hand and the hour hand when the passenger will board the next train?

6. Pranil works in an electronics goods shop and the shop has offered 30% discount on MRP on all the goods in the month of October. On top of it, the shop gives successive discount of 10% if the person uses e-payment mode. Pranil remembers that the maximum sale occurred after 17th October but before 21st October while his colleague remembers that the maximum sale happened after 19th October but before 24th October. Owner listens to both of them and concludes a date which is precisely common. What is the date as per your point of view?

7. a) In a day, how many times is a straight angle formed between minute and hour hands?

   b) In a day, how many times is a right angle formed between minute and hour hands?

8. Find the angle between the minute hand and the hour hand at 7: 40 pm?

9. By 20 minutes past 5, how many degrees has the hour hand has turned through?

10. At what time between 3′o clock and 4′o clock are the hands of a clock three degrees apart?

11. Indian government has announced a complete lockdown for entire country from the Midnight of 25th March 2020 till the midnight of 14th April 2020 due to pandemic disease Covid-19 outbreak. But keeping in view the grim situation, the lockdown was further extended till 3rd May 2020.

a) Calculate the total number of days for which the lockdown lasted.

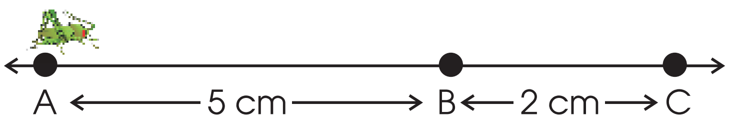
b) If 25th March 2020 is Wednesday, then find which day of the week will fall on 3rd May 2020.

12. India is 9 hours and 30 minutes ahead of Ottawa ON, Canada. What is time in Canada when it is 1:25 am in India?

2.4 Time, Work And Distance

Introductory Problem

An insect moves from point A to point C and returns to point B taking a total time of 6 seconds.



a) What is the total distance travelled by the insert?

b) Calculate the speed of insect?

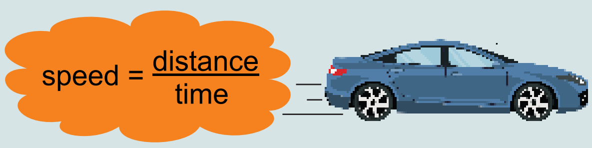
Relation Of Speed , Distance And Time

Speed is inversely proportional to time (if distance is constant) i.e. When time increases, speed decreases and vice versa.

EXAMPLE: 11

Speeds of two cars, A and B is in the ratio

$$\frac{Speed\ of\ car\ A}{Speed\ of\ car\ B} = \frac{x}{y}$$



$$\frac{Time\ taken\ by\ car\ A}{Time\ taken\ by\ car\ B} = \frac{x}{y}$$

$$\frac{Time\ taken\ by\ car\ B}{Time\ taken\ by\ car\ A} = \frac{x}{y}$$

Time taken by Car A: Car B= y : x

Work And Time

The physical or mental effort directed towards doing something refers to the work. In simple language work is an outcome of an effort applied for a particular time.

Work is directly proportional to time. i.e. more time given for work will increase the work. If amount of work i.e. a task remains constant, then two things play a vital role in the condition

1. Number of persons working
2. Number of hours

If In a task, the number of persons increase then the time decrease and vice versa

Relation Of Time And Work Done

Ravi does a work (W) for *n* days

then work done by Ravi in 1 day = $1/n^{th}$ of W = W/n

Nitish does same work (W) for *m* days

Then, work done by Nitish in 1 day = $1/m^{th}$ of W = W/m

Ravi and Nitish started working together and finished the work in *p* days.

Work finished by both in 1 day = $1/p^{th}$ of W

Therefore, $\frac{W}{n}+\frac{W}{m}=\frac{W}{p}$

So result can be generalized as $\frac{1}{n}+\frac{1}{m}=\frac{1}{p}$ .... (1)

Hence total no. of days taken by both of them:

$$\boxed{p=\frac{mn}{m+n}} \quad ...... (2)$$

If more than two persons work together for same task than result becomes:

$$\boxed{\frac{1}{n_1}+\frac{1}{n_2}+\ldots+\frac{1}{n_a}=\frac{1}{p}}$$

(n_a represents number of days taken by a^{th} person)

Example 12:

Reena completes her canvas painting in 4 days. Mihir finish the same canvas painting in 5 days. If they both work together find out the number of days taken by them to finish it.

No. of days taken together = $\frac{4\times 5}{4+5}=\frac{20}{9}$ days

= $2\frac{2}{9}$ days

Negative Work:

If two objects or persons are working against each other due to which a delay or incompletion in task occurs is referred as negative work.

Example : 13

Pipe A can fill a tank in 20 hours. But due to a leakage it is taking thrice the time to fill the tank. If the tank is full, how long will it take to drain out the tank, If the pipe A is closed?

Let Initial Time taken by pipe A = x hours

x = 20 hours

Tank filled in 1 hour = $\frac{1}{x}$

After leakage time taken by pipe = 3 x

$\Rightarrow$ Water filled in tank after leakage (in 1 hour) = $\frac{1}{3x}$

Let the tank get emptied in p hours

Water flows out in 1 hour = 1/p

Amount of water flows in 1 hour = water left in tank due to leakage

$$\frac{1}{p}=\frac{1}{x}-\frac{1}{3x}$$

$$\frac{1}{p}=\frac{2}{3x}$$

$$p=\frac{3x}{2}\text{ hours}=\frac{3}{2}\times 20=30\text{ hours}$$

$\Rightarrow$ Tank is drained out in 30 hours

Exercise: 1C

1. Navya is twice as efficient as Nitti. If they take10 days to finish a certain job together. How much time will they take individually to finish the same job?

2. A piece of work is finished in 30 days by A. Since C is thrice as good as A andA is twice as good as B. If A, B, C work together, then how many days will they take to finish the work?

3. X can do a piece of work in 60 days, whereas Y can do the same work in 40 days. Both started the work together, but X left after 10 days before the completion of work. Find how many days will it take to complete the work?

4. A train is running at 7/11 of its own speed due to fog and reached a place in 44 hours. What was the original time taken by the train if it runs at its own speed?

5. a) The signal poles on a railroad are placed 100 m apart, how many poles will be passed by a train in 8hours if the speed of the train is 45 km/h.

   b) If 7201 poles are to be installed within two stations at equal distance covering distance of 360km, Find out the distance between two consecutive poles.

6. 100 persons begin to work together on a project which was expected to be completed in 40 days. But after few days 40 persons left. As a result, the project got delayed by 10 days. How many days after the commencement of the project did the 40 persons left?

7. The efficiency of x,y,z are in ratio of 3:2:6 to finish a task. If they work together, they can finish it in 2 hours; find the time taken by them if they do the task individually?

8. A is three times as efficient as B. Also, A takes 30 days less than B for doing a piece of work. Find the time taken by them if they work a) individually b) together?

9. Machine P is 40% more efficient than Machine Q. Machine P can make 100 bags alone in 30 hours. Find the time taken to complete the order if both the machines work together?

10. A Corporate firm is running in loss, they had categorized their employees into two teams, Team A and Team B. The firm has funds to pay Team A for 21 days only. If they work with Team B then the firm can pay for 28 days only. Management has decided that they will allow both the teams to work for as long as they are having enough funds to pay them. Find out for how many days both the teams could work together?

2.5 Mensuration

Mensuration is a branch of Mathematics which deals with area, perimeter, volume of geometric figures

Introductory Discussion

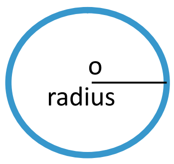
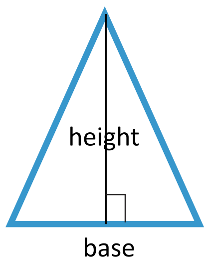
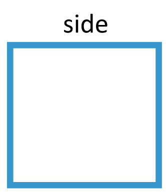
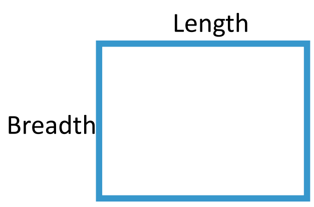
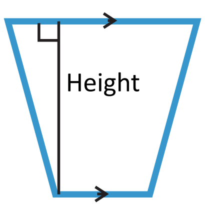
A restaurant sells paranthas in three different sizes. Paranthas are circular in shape. The small parantha is 15 cm in diameter and costs Rs 49. The medium sized parantha is 30 cm in diameter and costs Rs 79. The king sized parantha of diameter 35 cm costs Rs 99. Which of the paranthas is worth purchasing?

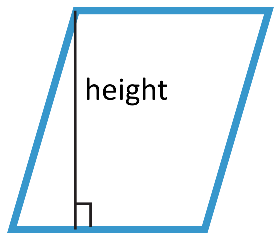
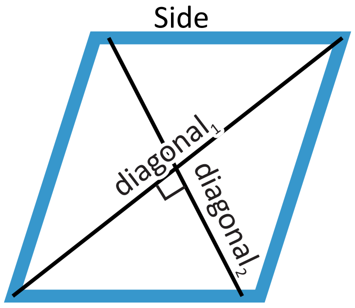
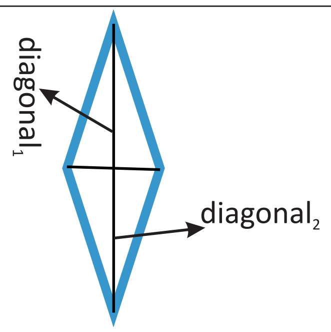
Concept of 2D And 3D Shapes:

A two dimensional figures is a Nane figure having length and breadth . It has no thickness. Example: Square, triangle, circle etc.

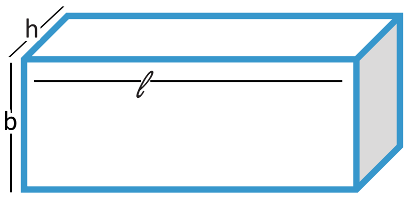
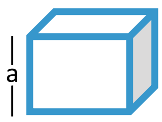
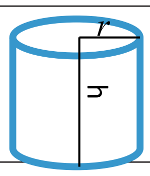
A three dimensional solid has three dimensions length, breadth and height. Example: Cuboid, Cube cylinder, cone etc

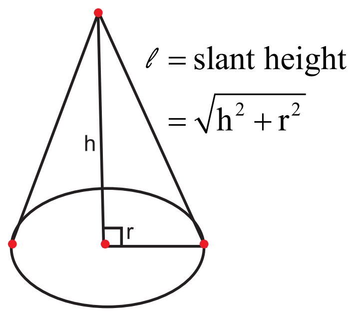
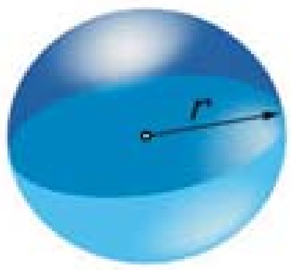
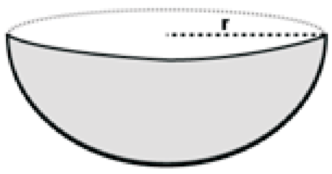
Let's recall formulas for some 2D Shapes

NAME	SHAPE	PERIMETER	AREA
CIRCLE		$2\pi r$	πr^2
TRIANGLE		Sum of all sides	If base *b* and corresponding height *h* are given, then area= $\frac{1}{2} \times base \times corresponding\ height$ If three sides a, b, c are given then $s = \frac{a+b+c}{2}$ and Area= $\sqrt{s(s-a)(s-b)(s-c)}$
SQUARE		4 X side	Side X Side
RECTANGLE		2 (length + breadth)	Length X breadth
TRAPEZIUM		Sum of all sides	$\frac{1}{2} \times height \times$ $(sum\ of\ parallel\ sides)$

PARALLELO-GRAM		2 x sum of adjacent sides)	Base X corresponding height
RHOMBUS		4 X side	$\frac{1}{2} \times diagonal_1 \times diagonal_2$
KITE		Sum of all sides	$\frac{1}{2} \times diagonal_1 \times diagonal_2$

Let's Recall Formulas for some 3D figures

NAME	FIGURE	LATERAL CURVED SURFACE AREA	TOTAL SURFACE AREA	VOLUME
CUBOID		2hx(l + b)	2(lb + bh +hl)	l X b X h
CUBE		$4a^2$	$6a^2$	a^3
CYLINDER		$2\pi rh$	$2\pi r(r+h)$	$\pi r^2 h$

CONE		$\pi r \ell$	$\pi r(r+l)$	$\frac{1}{3}\pi r^2 h$
SPHERE		$4\pi r^2$	$4\pi r^2$	$\frac{4}{3}\pi r^3$
HEMISPHERE		$2\pi r^2$	$3\pi r^2$	$\frac{2}{3}\pi r^3$

FUN FACT

Conversion of Solids

If a solid is melted to form a new solid, then the volume will remain same in both cases.

Length of Longest Rod :

Length of longest rod that can be fitted in a cuboid $=\sqrt{L^2+B^2+H^2}$

Where L = length, B= breadth, H =height

Length of longest rod that can be fitted in a cube $=\sqrt{3}a$

Where a = side of a cube

Exercise: 1 D

1. A cylindrical tank of base diameter 16 m and height 6 m requires a non-porous lining on its circular base and curved walls. The lining of the base costs Rs . 14. 20 per m^2 and on the curved walls it costs Rs 16.30 per square meter

   a) Find the area of the base.

   b) Find the total area of lining.

   c) Determine the total cost of lining.

2. The area of a rectangular field is 104000 m^2. This rectangular area has been drawn on a map to the scale of 1cm to 100 m. The length is shown as 7. 50 cm on the map.

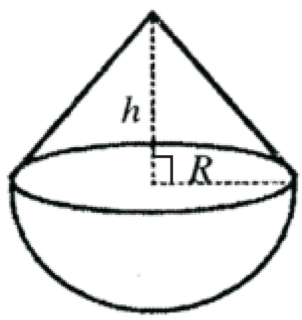
   a) Find the actual breadth of the rectangular field.

   b) Find the perimeter of rectangular field represented on map.

3. A rope by which a mare is tied at a centre of the circular field is decreased from 23m to 11 m.

   a) What is the decrease in area more will be able to graze now?

   b) Find the percentage decrease in area grazed? due to decreased length of rope.



4. A cone is surmounted on a hemisphere as shown here. The radius of the solid formed is 6 cm and its height is 21 cm. Find:

i) Curved surface area of the solid

ii) Volume of the solid.

5. Find the area of the greatest circle which can be inscribed in a square whose perimeter is 180 cm?

6. The length and breadth of rectangle are increased by 30% and decreased by 20 % respectively. Determine the percentage of increase or decrease in its area.

7. Dimensions of the cuboidal hall are 10 m x 12 m x 15 m. Find the length of the longest rod that can be fitted into the hall.

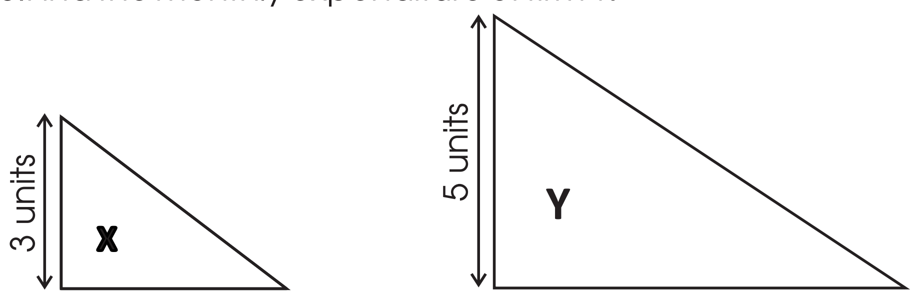
8. A rectangular hall is to be carpeted with a vinyl flooring sheet 25 cm wide, available at the rate of Rs. 22.60 per metre square. If the area of the hall is 117 m^2, then

   a) Find the length of vinyl flooring sheet required.

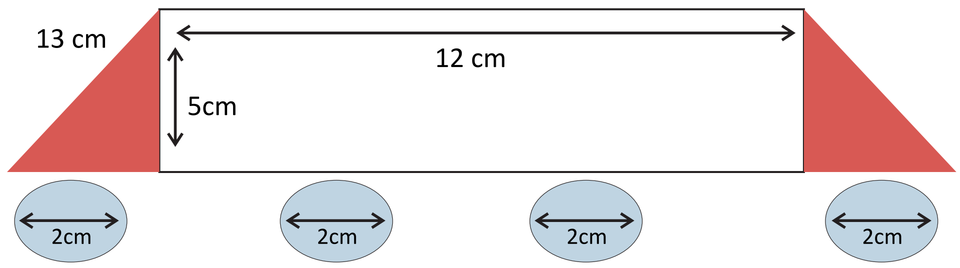
   b) Calculate the cost of vinyl sheet.

9. Area of circle is numerically equal to twice of its circumference. Find the diameter of the circle.

10. The following isosceles right-angled triangles represent the monthly expenditures of two law firms X and Y. If the monthly expenditure of firm X is Rs. 63000.Find the monthly expenditure of firm Y.



11. Shakuntala designed a toy car. Help her to find the area of the given below.



(Both the triangles are identical)

12. A circular park 120 m in diameter has a path of 10m wide all around outside, which is to be gravelled. The MCD received tenders from three companies as follows:

COMPANY	A	B	C
COST OF GRAVELLING (in Rs)	120 per m^2	140 per m^2	10000 per $100m^2$

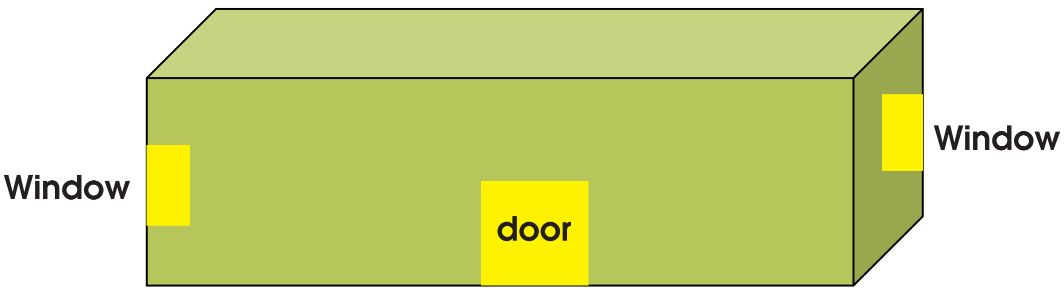
a) Find the area of path.

b) Which company will get the tender? (Hint: MCD will be selecting the tender of the company which has quoted the least)

c) Calculate the total amount paid by MCD to the selected company.

13. A swimming pool 36 m wide and 27 m long is 3m deep on the shallow end and 4 m deep at the deeper end. Find the capacity of the pool (in litres).

14. Nishant owns a house of dimensions 10 feet x 8 feet x 9 feet. His house has two entry doors on the opposite sides, of dimension 7 feet by 3 feet each and two windows on opposite walls of dimensions 3 ft x 2 ft each. He wants wall papering to be done in his house. If the wallpaper that he selects is 2 ft wide and the cost is Rs. 25/ ft. Find the total cost paid by Nishant, if the labour charges Rs. 10/ sq feet.



15. A cubical gold block of dimensions of 9cm x 11cm x 12 cm is melted and recasted into spherical balls of radius 3 mm. Find the number of spherical balls formed.

2.6 Seating Arrangements

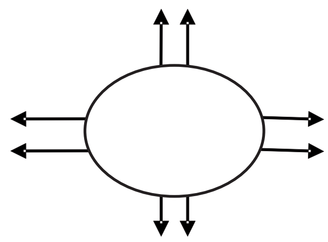
Introductory Discussion

Eight students A, B, C, D, E, F, G and H participated in a game in which they are seated on a circle., While A,B,C,D are seated on odd number. of seats respectively, E,F,G and H are seated on even number respectively.

Try all the ways of arranging their seating and answer the following questions accordingly.

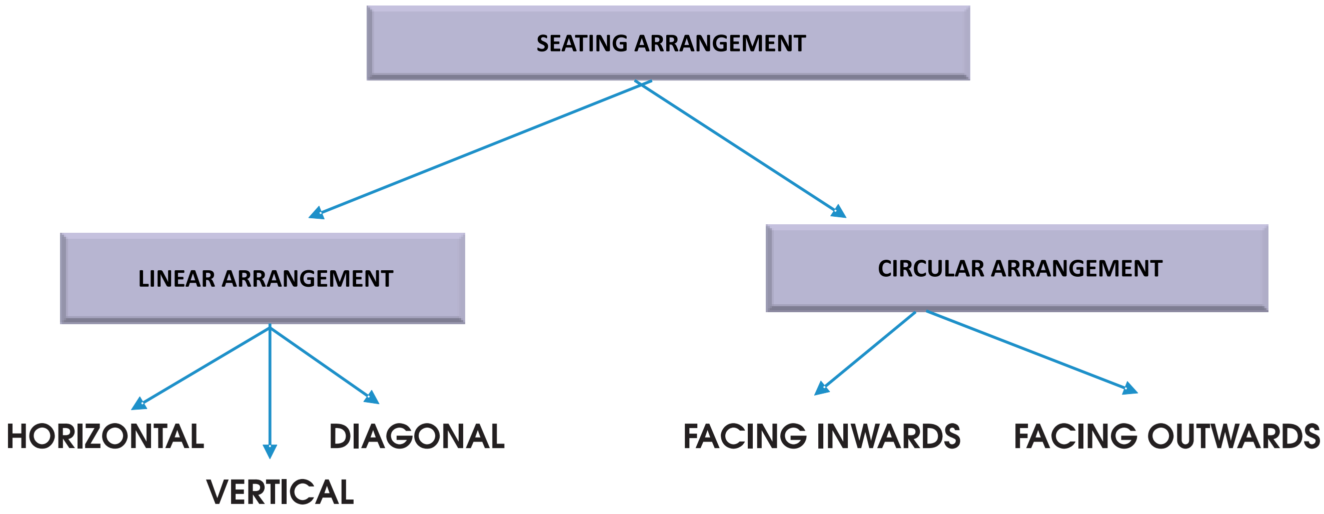
a) Name the students seating on 4th and 8th position

b) Name the student who are adjacent to G



The concept of arrangement plays a very important role in making our life simpler and easier. In this topic we will be dealing with the seating arrangements done under certain condition(s).

Arrangement is a plan which specifies how a group of objects/ persons are placed/allocated their positions in a particular situation/framework. It represents the patterns which may be linear or circular.



Linear Arrangement

Logical arrangement of any objects/persons; either horizontally or vertically or diagonally is referred as linear arrangement.

Series connection of resistors is an example of horizontal linear arrangement.
Students sitting in classroom or examination hall are an example of vertical linear arrangement.

Circular Arrangement :

If objects/persons are arranged in a circular manner than the arrangement is known as a circular arrangement.

A. FACING INWARDS (TOWARDS CENTRE): Objects or people in an arrangement, facing towards the center of circle. Example - Round table conference.

B. FACING OUTWARDS(AWAY FROM CENTRE):Every object or person in an arrangement, facing outwards. Example: Musical Chair

Seating Arrangement In A Photograph:

When a photographer is clicking a photograph, the right and the left references change i.e. the viewer's right hand side becomes photographer's left hand side and vice-versa.

Example : 14

Five children are sitting in a row. Shiva is sitting next to Priya but not Diya. Kavya is sitting next to Riya who is sitting on extreme left and Diya is not sitting next to Kavya. Who are adjacent neighbours of Shiva.

RIYA KAVYA PRIYA SHIVA DIYA

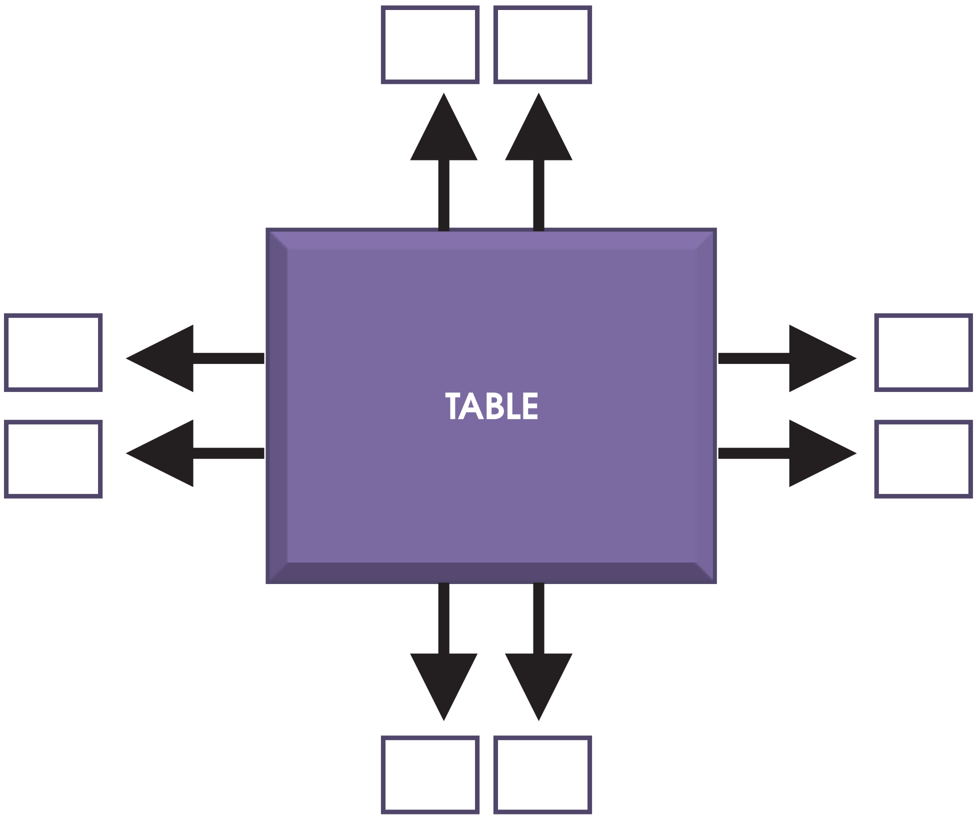
Extreme left Next to Riya Shiva is sitting next to Priya but not to Diya

Hence Priya and Diya are adjacent neighbors of Shiva.

Exercise 1E

1. Six students P,Q,R,S,T and U are sitting in two rows, three are in each row. T is not at the end of any row. S is second to the left of U. R is the neighbour of T,Q is the neighbour of U. Arrange the members of two rows. Who is sitting diagonally opposite to Q?

2. Group of 8 persons A,B,C,D,E,F,G and H are seated around a square table while facing towards each other ,two persons are seated on each side. There are three ladies in a group and they are not allowed to sit next to each other. F is seated between H and B, also C is seated between E and B. D is a lady member who is seated second to the left of F. A is a lady member seated opposite to B. B is a male member. A lady member must be seated in between B and E.



Answer the following questions on the basis of the above information:

i) Identify the lady members.

ii) Name the members who is seated immediate left to B

iii) Name the members who are adjacent to D

3. Riya is known for her good photography skill and has won many prizes. On the occasion of family function five cousins approached her for clicking their photograph. Riya had them all sit on a bench. Siya is seated left to Rani and to the right of Bubbly. Maria is seated to the right of Rani, Reet is between Rani and Maria.

   a) Who is seated in the middle

   b) Who is seated immediate right to Reet.

4. In an exhibition seven different electronic companies are displaying their new Air Conditioner Model on seven different tables - LG, LLOYD, VOLTAS, HITACHI, DECCAN, SAMSUNG, and WHIRLPOOL. All the models are facing towards east. LG is next to the right of WHIRLPOOL, WHIRLPOOL is placed

fourth to the right of VOLTAS. HITACHI is placed between LLOYD and SAMSUNG. VOLTAS which is placed third to the left of LLOYD, is at one end.

a) Write the arrangement of companies of AC from the left to the right.

b) Which AC brand lies between HITACHI and WHIRLPOOL.

c) Name the air conditioner that is placed at the extreme right.

5. Three Doctors Dr. Aggarwal, Dr.Chabbra and Dr. Roy are available in LIFE hospital of New Delhi. Dr. Aggarwal is available in hospital between 1 pm and 5 pm on Monday, Wednesday and Sunday. Dr. Chabbra is available in hospital between 11 am and 3 pm on Tuesday, Wednesday, Friday and Sunday. Dr. Roy is available between 10 am and 1 pm on Tuesday and Thursday and also available on Friday, Saturday and Sunday between 3 pm and 5 pm.

a) On which day are all three doctors available in the hospital.

b) Ravi wants to consult two doctors Dr. Chabbra and Dr. Roy on the same day. Workout suitable day and time for him to visit the hospital.

6. Five friends are standing in a line and facing towards the wall wearing Red, Grey, Yellow, Violet and Black Shoes. The persons wearing Yellow and Red Shoes are not standing at the end position. The person in the middle position is wearing Black shoes and the person with Red Shoes is not in his left. The person wearing violet shoes is standing on extreme right.

a) Who is on 4th position from right?

b) Who is standing on the extreme left position?

7. Six persons P, Q ,R,S,T and U are sitting in a circle with their faces towards the centre. S is on the immediate left of T, P is on the left of S and U is immediate neighbor of T. R is sitting second to the right of U..

a) Name the immediate neighbours of Q.

b) Who is between S and R?

Summary

1. AVERAGE (MEAN) = $\dfrac{Sum\ of\ all\ observations}{total\ number\ of\ observations}$

2. Weighted Average is the mean of a set of numbers in which some elements of the given data carry more weightage than others.

$$\bar{x}_w = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

3. Average speed is independent of direction and average velocity is dependent of direction.

4. Average speed cannot be zero but average velocity can be zero.

5. **Calendar** is collection of days, weeks and months in a tabular form for a specific year.

6. **Odd Days** are the number of days left after exact number of weeks.

7. A century which is a leap year also has zero number of odd days.

8. Angular value of a minute hand is 6^0.

9. Speed of an hour hand = ½ degree per minute

10. Speed of a minute hand is 6^0 per minute

11. Two hands of clock minute hand and an hour hand collides each other after every $65\frac{5}{11}$ minutes.

12. Time and angle formed between minute and hour hand is related as

    i) $t = \dfrac{2}{11}(H \times 30^0 + A)$

    ii) $A = 30 \times H \dfrac{11}{2} \times t$ (t = time in minutes indicated by minute hand)

Note: Value of A is always taken as positive, negative sign is ignored.

13. Volume of a solid = Area of base X height of object

Chapter - III

Set

Learning Objectives:

After the completion of this section, the students will be able to:

- Define sets and type of sets
- Draw venn diagrams and do operation on sets

Concept Map

What is a Set?

Representation:
- Roster form
- Tabular form

Types of Sets:
- Empty Set
- Finite Set
- Infinity Set
- Subset
- Proper Set
- Universal Set

Intervals as Subsets of:
- Open interval
- Closed interval
- Semi open/ closed interval

Operations on sets:
- Union of Sets
- Intersection of sets
- Complement of a Act
- De Mogan's Law

Venn Diagrams:
- Practical Problems
- Solved using
- Venn Diagrams

3.1 Introduction

In everyday life we speak of collections - whether it be collection of books written by great poet Munshi Prem Chand, or it be collection of movies featuring great actors like Amitabh Bachchan or it be collection of melodious songs by Nightingale of India, Lata Mangeshkar or collection of months in a year and the list is endless.

Similarly, most of us listen to music but not everyone has same taste of songs. Rock songs are separated from classical or any other genre, where particular songs belong to that particular genre.

Consider one more real life situation. You may have noticed that every school has some set of rules that need to be followed by every student and employee of the school. These may include disciplinary rules, timing rules, rules for leave etc. Hence all different types of rules are separated from each other, thus forming different categories or sets.

In Mathematics too, we come across collection of numbers, functions etc., for example, natural numbers, integers, trigonometric functions, algebraic functions and so on.

To learn sets we often talk about such collection of objects. The concept of sets is used in the foundation of various topics in Mathematics.

Now, the above collection of objects or numbers or functions is well-defined as we can identify and decide whether a particular object belongs to the collection or not.

3.1 What is a Set?

Definition 1 : A set is a **well-defined** collection of **distinct** objects.

The word 'well-defined' refers to a specific property or some definite rule on the basis of which it is easy to identify and decide whether the given object belongs to the set or not. The word `distinct' implies that the objects of the set must be all different.

For example, consider the collection of all the students in your class whose names begin with letter 'S'. Then this collection represents a set as, the rule of the names beginning with 'S' has been clearly specified.

However, the collection of all the intelligent students in your class does not represent a set because the word 'intelligent' is relative and degree of intelligence of a student may be measured in different spheres. Whosoever may appear intelligent to one person may not appear the same to another person.

Illustration 1: Which of the following collections is not a set? Give reason for your answers.

(i) The collection of all boys of your class.

(ii) The collection of all monuments in Delhi made by Mughal emperor Akbar.

(iii) The collection of all the rivers in Delhi and Uttar Pradesh.

(iv) The collection of most talented singers of India.

(v) The collection of measures of central tendency of a given data.

(vi) The solution of the equation $x^2 - 6x + 8 = 0$

Solution: (iv)

This is because it is not a well-defined collection as the criteria for determining the talented singer may vary from person to person.

In (i), (ii), (iii), (v) and (vi) we can list the objects belonging to the given collection.

For example, in part (vi), $x = 4$ and $x = 2$ belong to the collection which satisfy the given equation $x^2 - 6x + 8 = 0$,

In fact, no real number except $x = 2, 4$ belongs to this collection.

Note:

1. The objects that belongs to a particular collection or set are also known as its members or elements.
2. Sets are usually denoted by capital letters, say, A, B, C, X, Y etc.

3.2 Representation of Sets

There are two methods of representing a set:

1. Roster form or Tabular form
2. Set builder form

Let us discuss both of them in detail.

1. Roster Form or Tabular Form

In Roster form, all the elements of a set are listed, separated by commas and enclosed within curly braces { }.

For example, the set of vowels of English Alphabet may be described in roster form as:

$$V = \{a, e, i, o, u\}$$

Since 'a' is an element of a set A, we say that `-"a belongs to A"' and the Greek symbol '$\in$' (epsilon) is used to denote the phrase 'belongs to'.

Thus, mathematically, we may write the above statement as:

$$a \in A$$

Similarly, $e \in A$, $i \in A$, $o \in A$, and $u \in A$.

Also, since 'b' is not an element of set A, we write, $b \notin A$ and is read as 'b does not belong to A'.

Let us consider some more examples of representing a set in roster form:

(a) The set of all even natural numbers less than 10 may be written as:

$A = \{2, 4, 6, 8\}$

(b) The set of letters forming the word 'NUMBERS' is B = {N, U, M, B, E, R, S}

Note 1: While writing the set in roster form, the element is not repeated, that is, all the elements in the set are distinct. Please note that the repetition of the elements of a set does not alter the set.

For example, the set of letters of the word 'FOLLOW' is written as:

$$A = \{F, O, L, W\}$$

Interestingly, when we consider the set of letters of the word 'WOLF', it is given by B = {W, O, L, F}. Please note that the elements of set A are exactly the same as set B. Such are called equal sets, which would be discussed later. Moreover, a set of particular alphabets may form different words, as noticed above.

Similarly, the set of alphabets of the word, 'ABLE' is given by X = {A, B, L, E} and the set of alphabets of the word 'LABEL' is also Y = {A, B, L, E}.

So, we may understand from this now that the set comprising of different alphabets may form different words, depending on the repetition of alphabets in the words. Moreover, the elements in the set may be written in any order. So, X = {A,B,L,E} is equal to the set Z = {B,E,A,L}.

2. Set-Builder Form

In this form, a set is described by a characterising property of its elements.

For example, in the set A = {1, 2, 3, 4, 5}, all the elements possess a common property, that is, each of them is a natural number less than 6. So, this set can be written as:

A = {x : x is a natural number and $x < 6$} and is read as "the set of all 'x' such that x is a natural number and x is less than 6."

Hence, the numbers 1, 2, 3, 4 and 5 are the elements of the set A.

Illustrations 2: Write the following sets in roster form:

(i) A = {x : x is a letter in the word 'MATHEMATICS'}

(ii) B = {x : x is a letter in the word 'STATISTICS'}

(iii) C = {x : x is a two digit number such that the product of its digits is 12}

(iv) D = {x : x is a solution of the equations $x^2 - 4 = 0$}

Solution:

(i) A = {M, A, T, H, E, I, C, S}

(ii) B = {S, T, A, I, C}

(iii) C = {26, 34, 43, 62}

(iv) D = {–2, 2}

Illustrations 3: Write the following sets in the set-builder form:

(i) A = {1, 4, 9, 16}

(ii) B = {2, 3, 5, 7, 11}

(iii) C = {–2, 0, 2}

Solution:

(i) $A = \{x : x = n^2, \text{ where } n = 1, 2, 3, 4\}$

(ii) $B = \{x : x$ is a prime number less than $12\}$

(iii) $C = \{x : x$ is a solution of the equation $x^3 - 4x = 0\}$

Note: The symbols for special sets used particularly in mathematics and referred to throughout, are given as follow:

N : the set of all natural numbers

Z : the set of all integers

W : the set of all whole numbers

Q : the set of all rational numbers

R : the set of all real numbers

Z^+ : the set of all positive integers

Q^+ : the set of all positive rational numbers

R^+ : the set of positive real numbers

Check your progress 3.1

Q.1 Which of the following are sets?

(i) The collection of most talented authors of India.

(ii) The collection of all months of a year beginning with letter M.

(iii) The collection of all integers from –2 to 20.

(iv) The collection of all even natural numbers.

(v) The collection of best tennis players of the world.

Q.2 Write the following in set-builder form:

(i) $A = \{4, 8, 12, 16, 20\}$

(ii) $B = \{2, 3, 5, 7, 11, 13, 17, 19, ...\}$

(iii) $C = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$

(iv) $D = \{-1, 1\}$

(v) E = {41, 43, 47}

(vi) F = $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$

(vii) G = $\left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, ...\right\}$

Q.3 Write the following in roster form:

(i) A = {x : x is a whole number less than 5}

(ii) B = {x : x is an integer and $-4 < x \leq 6$}

(iii) C = The set of all letters of the word FOLLOW.

(iv) D = {x : x is a two-digit number such that the sum of its digits is 6}

(v) E = the set of all letters of the word 'ARITHMETIC'

Q.4 Match each of the sets on the left expressed in roster form with the same set described in the set builder form on the right:

(i) {1, 2, 3, 6, 12}	(a) {x : x is a day of the week beginning with T}
(ii) {2, 3, 5}	(b) { x : x = $n^2 - 1$, $n \in N$ and $n \leq 4$}
(iii) {Tuesday, Thursday}	(c) {x : x is a factor of 12}
(iv) {0, 3, 8, 15}	(d) {x : x is the smallest natural number}
(v) {1}	(e) {x : x is a prime number less than 7}

Q.5 Let A = {1, 2, 3, 4, 5}, B = {x : x is a factor of 4}, C = {1, 4, 9}. Insert the correct symbol '$\in$' or '$\notin$' in each of the following to make the statement true

(i) 4 _______ A

(ii) 3 _______ B

(iii) 9 _______ C

(iv) 1 _______ B

(v) 3 _______ C

3.3 Types of Sets

Consider the following sets:

(i) A = {x : x is an even prime number}

(ii) B = {x : x is an even prime number greater than 2}

(iii) C = {x : x is a prime number less than 50}

(iv) D = {x : x is a prime number}

Now, in case (i), we observe that there exists only one even prime number which is 2. Therefore, the set A contains only one element. Such a set which consists of only single element is called a singleton set.

Definition 2: Singleton Set

A set consisting of a single element is called a singleton set.

Examples:

(a) Let X = {x : 1 < x < 3 and x is a natural number}. This is a singleton set as the only element of set B is '2'.

(b) Let Q = {x : x is the day of the week starting with alphabet 'M'}. This is also a singleton set as the only day of the week starting with alphabet 'M' is Monday.

Consider case (ii). We observe that there does not exist any prime number which is greater than 2. Therefore, this set B does not contain any element. Such a set is called an empty set or a void set.

Definition 3: Empty Set

A set which does not contain any element is called the empty set or null set or the void set and is denoted by the symbol ø or { }.

Examples:

(a) Let A = {x : x is a month of the year starting with alphabet B}. This is an empty set as no month of the year in English calendar starts with alphabet 'B'.

(b) Let A = {x : x is a real solution of the equation $x^2 + 2 = 0$}. This set is an empty set as there is no real solution of the quadratic equation $x^2 + 2 = 0$.

In case (iii), we observe that there are fifteen elements, which is finite in number. Such a type of set is called a finite set.

Definition 4: Finite Set

A set which is either a null set or whose elements can be numbered from 1 to n for some positive integer n is called a finite set.

In other words, a finite set is either an empty set or has definite number of elements.

The number 'n' is called the cardinal number or order of the finite set and is denoted by n (A).

Note: The cardinality of an empty set is zero, i.e., n (ϕ) = 0.

Examples:

(a) Let X = {x : x is month of the year beginning with letter J}

= {January, June, July}

There are three elements is the set x, given above. Therefore, the set X is a finite set and n (x) = 3.

(b) Let Y = {x : $-1 \leq x \leq 4$ and x is an integer}.

= {−1, 0, 1, 2, 3, 4}

There are six elements is this set Y. Therefore, set Y is a finite set and n (Y) = 6.

Now we consider case (iv). The number of elements in this set is infinite and moreover, the elements of this set cannot be listed by positive integers. Such a set is called an infinite set.

Examples:

(a) The set of natural numbers, N = {1, 2, 3, 4, ...} is an infinite set.

(b) The set of even natural numbers, say, E = {2, 4, 6, 8, ...} is an infinite set.

Note: All infinite sets cannot be described in roster form. For example, the set of real numbers cannot be described in this form as between any two given real numbers, there exists infinite number of real numbers. So, it is not possible to list them in roster form.

Illustration 4: State which of one following sets are finite and which are infinite.

(i) $\{x : x \in Z$ and satisfies the equation $(x+1)(x+2) = 0\}$

(ii) $\{x : x \in N$ and $x^2 - 1 = 3\}$

(iii) $\{x : x$ is an odd natural number$\}$

(iv) $\{x : x$ is an integer and $-1 < x < 0\}$

(v) $\{x : x$ is an integer less than $0\}$

(vi) $\{x : x$ satisfies the identity $\cos^2x + \sin^2x = 1\}$

Solution:

(i) $A = \{-1, -2\}$, is a finite set as it consists of only two elements.

(ii) $B = \{2\}$, is a finite set having only one element.

(iii) $C = \{1, 3, 5, 7, 9, ...\}$ is an infinite set as it doesn't have a definite number of elements.

(iv) ϕ, which is a finite set.

(v) $D = \{....., -3, -2, -1\}$ is an infinite set as all its elements cannot be listed.

(vi) There are infinite real numbers that satisfy the identity $\cos^2x + \sin^2x = 1$. Therefore, it is an infinite set.

3.4 Subsets

Consider the following sets:

E = set of all students in your school.

F = set of all students in your school who have taken up commerce as their subject stream.

We note that every element (student) in set F is a student of your school itself, which is described by set E.

In other words, every element of F is also an element of E. The set F is said to be a subset of E and in symbols it is expressed as $F \subseteq E$. The symbol '$\subseteq$' stands for 'is a subset of' or 'is contained in'.

Definition 5 : Subset

Let A and B be two sets. If every element of set A is an element of set B also, then A is called a subset of B and is written as $A \subseteq B$.

In other words, $A \subseteq B$, if whenever $a \in A$, then $a \in B$. Using the symbol '$\Rightarrow$' which means 'implies', we can write the definition of subset as:

$A \subseteq B$ if $a \in A \Rightarrow a \in B$

If A is not a subset of B, we write $A \not\subseteq B$

Note:

1. The empty set is a subset of every set.
2. Every set is a subset of itself.
3. The symbol '$\subseteq$' suggests that either $A \subseteq B$ or $A = B$. $A \subset B$ implies that A is a proper subset of B, that is,

   A has lesser number of elements than B.

   A = B implies that A and B are equal sets, that is, they have exactly the same elements.

   In other words, two sets are said to be equal if every element of A is a member of B and every element of B is an element of A.

   That is, to say, $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

Illustration 5: Consider the sets ϕ; $A = \{1, 2, 3\}$; $B = \{1, 4, 5\}$; $C = \{1, 2, 3, 4\ 5\}$; $D = \{x : x \in N \text{ and } x < 4\}$.

Insert the symbol '$\subseteq$', '$\not\subseteq$' or '=' between the following pair of sets:

(i) $\emptyset$ __ B

(ii) A ______ B

(iii) B ______ C

(iv) A ______ D

Solutions:

(i) $\phi \subseteq B$ as ϕ, the empty set is a subset of every set.

(ii) $A \not\subseteq B$ as $2 \in A$ but $2 \notin B$.

(iii) $B \subseteq C$ as $1, 4, 5 \in B$ and they also belong to C.

(iv) A = D as sets A and D have exactly the same elements which are 1, 2 and 3.

Illustration 6: Let A {1, 2, 3, {4}, 5}. Which of the following are incorrect? Give reasons.

(i) $1 \in A$

(ii) $\{1, 2\} \subseteq A$

(iii) $\{1, 2, 4\} \subseteq A$

(iv) $\{4\} \in A$

(v) $\phi \subseteq A$

(vi) $\{4\} \subseteq A$

Solutions:

(i) Correct. This is because 1 is an element of A. Therefore, $1 \in A$.

(ii) Correct. Every element belonging to the set {1, 2} also belongs to A. Therefore, $\{1, 2\} \subseteq A$.

(iii) Incorrect. This is because, 4 is not an element of A but {4} is an element of A.

$\therefore \{1, 2, \{4\}\} \subseteq A$ but $\{1, 2, 4\} \not\subseteq A$

(iv) Correct. {4} is one of the elements of A. Therefore, $\{4\} \in A$.

(v) Correct. Empty set is a subset of every set.

(vi) Incorrect. This is because {4} is an element of set A. Therefore $\{\{4\}\} \subseteq A$ but $\{4\} \not\subseteq A$.

3.5 Power Set

Consider the set A = {1, 2, 3}

Now $\subseteq \phi$ A as empty set is a subset of every set. Also, A $\subseteq$ A, as every set is a subset of itself.

The other subsets of A are: {1}, {2}, {3}, {1, 2}, {2, 3}, {1, 3}.

The set of all these subsets of set A is called the power set of A.

Definition 6: Let A be set. Then, the set of all the subsets of A is called the power set of A and is denoted by P(A).

Thus, in the above example, if A = {1, 2, 3}, then,

P(A) = {ϕ {1}, {2}, {3}, {1, 2}, {2, 3}, {1, 3}, {1, 2, 3}}

Note: n (P(A)) = 2^3 = 8

Therefore, n (P(A)) = 2^n, where 'n' is the number of elements in A. In other words, the number of subsets of any set A are given by 2^n, where, 'n' is the number of elements in A.

Also, the number of proper subsets of A having 'n' elements is given by 2^n–1, which excludes the set itself.

3.6 Universal Set

A set that contains all the objects or elements in a given context is called the universal set and is denoted by U.

For example, let A = set of all even natural numbers and B = set of odd natural numbers. Then the universal set in this context, may be the set which contains all the elements, that is,

U = set of natural numbers

Universal set in this case can also be taken as whole numbers or integers or real numbers.

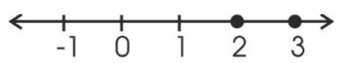
Note:

1. Universal set is the superset of all the sets under consideration.
2. $N \subseteq Z \subseteq Q \subseteq R$. This relation explains that every natural number is an integer, every integer can be put is the form of p/q, where $q \neq 0$, hence every integer is a rational number and all rational numbers are real numbers.

3.7 Intervals as Subset of the set of Real Numbers R

Consider the solution set of the equation $x^2 - 5x + 6 = 0$. In tabular form it is written as A = {2, 3}

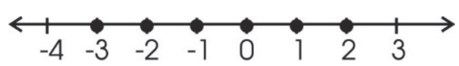
On the real number line it is represented as:



Let us consider another case.

Represent all the integers from –3, to 2, both in the roster form and graphically on the number line.

Now, in set notation form, it is given by A = {–3, –2, –1, 0, 1, 2} and plotting on the number line, we get,



Let's consider the following example of representing the real numbers, both as set notation and on the number line from –3 to 2.

The set of real numbers from –3 to 2 is denoted by $B = \{x : x \in R, -3 \leq x \leq 2\}$.

In this set it is impossible to list the real numbers, as between any two real numbers lie infinite real numbers, as real number cannot be separated by commas and listed. Therefore, in order to denote the above set we introduce the concept of intervals.

What is an Interval?

An interval is a set of real numbers between a given pair of real numbers, excluding or including one or both these real numbers.

It can be considered as a segment of the real number line, where in, the end points of the interval mark the end points of that segment. Let us now understand different types of intervals.

3.8 Types of Intervals

1. Closed Interval

Let a, b $\in$ R and a < b. Then the set of all real numbers from a to b, in the set builder form is written as A = $\{x : x \in R, a \leq x \leq b\}$.

In interval form, this set is represented as (a, b), indicating that it includes all real numbers form a to b, including the end points 'a' and 'b'. This type of interval which includes the end points in the set of real numbers between a given pair of real numbers is called a closed interval. The inclusion of the end points is indicated by square brackets ( ) in the interval notation.

Illustration 7: The interval notation for all real numbers from –3 to 2 is written as:

$(-3, 2) = \{x : x \in R, -3 \leq x \leq 2\}$

As a segment of the real number line, the representation is given by:

2. Open Interval

Let's now consider the set of real numbers between 'a' and 'b', where $a, b \in R$ and $a < b$.

This set of real numbers in set-builder form is written as:

$$B = \{x : a < x < b\}$$

In interval form, it is expressed as (a, b), indicating that it includes all real numbers between 'a' and 'b', excluding the end points 'a' and 'b'.

This type of interval which does not includes the end points in the set of real numbers is called an open interval and exclusion of the points is indicated by round brackets ( ) in the interval notation.

Illustration 8: The interval of real numbers between –3 and 2 is written as:

$$(-3, 2) = \{x : x \in R, -3 < x < 2\}$$

As a segment of the real number line, the representation is given by:

where the exclusion of end points –3 and 2 is illustrated by an open dot.

3. Semi-Open/Closed Interval

In the above cases, we had intervals where either both the end points are included or both excluded.

But there are intervals where either of the end points is included or excluded.

Consider the following set of real numbers: $A = \{x : a < x \leq b; a, b \in R \text{ and } a < b\}$

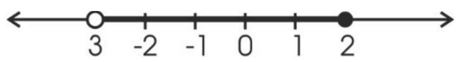
This set consists of all real numbers between 'a' and 'b' but 'b' is included in the set and 'a' is excluded from it. So, in interval notation, it is expressed as:

$$(a, b) = \{x : x \in R, a < x \leq b\}$$

Illustration 9: Consider the interval $(-3,2]$.

Now, $(-3,2]$ = {x : x $\in$ R,–3 < x $\leq$ 2} represents the set of real numbers between –3 and 2 where '2' is included in the interval but–3 is excluded from it.

As a segment on the real number line, the representation is given by:



Now consider the set of real numbers:

{x : a $\leq$ x < b; a , b $\in$ R, a < b}

This set consists of all real numbers between 'a' and 'b' but 'a' is included in the set and 'b' is excluded from it. So in interval rotation, it is expressed as:

$[a,b)$ = {x : x $\in$ R, a $\leq$ x < b}

Example: $(-3,2]$= {x : x $\in$ R, –3 $\leq$ x < 2} represents the set of real numbers between –3 and 2 where, –3 is included and 2 is excluded from the set.

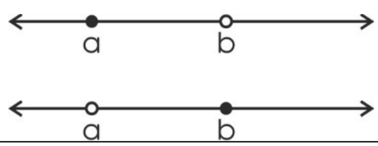
As a segment of the real number line, the representation is given by:

Note: The number (b – a) is called the length of the interval (a, b), (a, b), (a, b) or (a, b).

Illustration 10: The length of the interval (2, 7) is 5 and the length of the interval (–3, 4) is (4 – (–3)) = 7.

Let us consider brief summary of the context discussed pertaining to intervals.

Table 1.1.1

S. No.	Interval	Description	Graphical Representation
1.	Closed interval: (a, b)	{x : x $\in$ R, a $\leq$ x $\leq$ b}	
2.	Semi-open closed interval		

	(a, b) (a, b)	$\{x : x \in R, a \le x < b\}$ $\{x : x \in R, a < x \le b\}$	
3.	Open Interval: (a, b)	$\{x : x \in R, a < x < b\}$	
4.	(a, ∞)	$\{x : x \in R, x > a\}$	
5.	$(-\infty, b)$	$\{x : x \in R, x < b\}$	
6.	(a, ∞)	$\{x : x \in R, x \ge a\}$	
7.	$(-\infty, b)$	$\{x : x \in R, x \le b\}$	
8.	$(-\infty, \infty)$	R (real numbers)	

Check your progress 3.2

Q.1 Which of the following sets are finite and which are infinite? In case of finite sets, write its cardinality.

(i) $\{x : x$ is a natural number less than $100\}$.

(ii) The set of all prime numbers.

(iii) The set of the days of the week.

(iv) $\{x : x = n^2$, where n is a natural number$\}$.

(v) The set of all lines in a plane parallel to the line $2y = 3x + 7$.

(vi) $\{x : x$ is a real number and $0 < x < 1\}$.

Q.2 Which of the following sets are empty and which are singleton sets?

(i) $\{x : x$ is an even prime number$\}$.

(ii) $\{x : x$ is a natural number and $-1 < x < 1\}$.

(iii) $\{x : x$ is an integer and $-1 < x < 1\}$.

(iv) $\{x : x$ is a vowel in the word 'EYE'$\}$.

(v) $\{x : x + 10 = 0; x \in N\}$.

Q.3 Which of the following pairs of sets are equal? Give reasons.

(i) $A = \{-2, 3\}$, $B = \{x$ is a solution, of $x^2 - x - 6 = 0\}$

(ii) A = {x : x is a letter of the word 'FOLLOW'}

B = {y : y is a letter of the word 'WOLF'}

(iii) A = {x : x is a letter of the word 'ASSET'}

B = {y : y is a letter of the word 'EAST'}

(iv) A = {–1, 1}; B = {x : x is a real number satisfying the equation $x^2 + 1 = 0$}

(v) A = {1, 4, 9}; B = {x : x = n^2 where 'n' is a natural number less than 5}

Q.4 Let A = {1, 2, {3, 4}, 5}. Put the correct symbol in each of the following.

(i) {3, 4} _______ A

(ii) {1} _______ A

(iii) {3} _______ A

(iv) {{3, 4}} _______ A

(v) {1, 3} _______ A

(vi) {1, 5} _______ A

(vii) ϕ _______ A

Q.5 Write the following in set-builder form.

(i) (1, 3)

(ii) (–1, 3)

(iii) (–4, 0)

(iv) (–1, 1)

(v) (0, ∞)

Q.6 Let A = ϕ. Find P (P(A))

Q.7 Find the number of subsets of the set B = {a,b,c,d}.

3.9 Venn Diagrams

Venn diagrams are the pictorial representation of the relationships between sets. They are named after the English logician John Venn.

In Venn diagrams, the universal set is usually denoted by a rectangle and its subsets by circles or ellipses. It is denoted by U.

Illustration 10: Let U = {1, 2, 3, 4, 5} and A = {2, 4}. Represent the relationship between U and A using Venn diagram.

Solution: Set U = {1, 2, 3, 4, 5} is the universal set of which A = {2, 4} is a subset.

Therefore, in set notation we can express the relationship between A and U as $A \subseteq U$.

Using, Venn diagram, we can pictorially represent this relationship as given in Fig. 1.1.

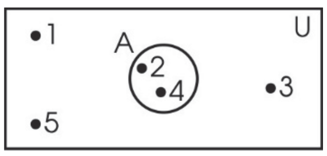


Fig. 1.1

Illustration 11: Let U = {1, 2, 3, 4, 5}, A = {1, 2}, B = {3, 5}. Represent the relationship between U, A and B using Venn diagrams.

Solution: Set U = {1, 2, 3, 4, 5} is the universal set. Sets A and B are subsets of u and we may also observe that A and B do not share any common element. Therefore, in set notation, we can express the relationship between A, B and U as: $A \subseteq U$ and $B \subseteq U$.

Therefore, pictorially, it can be represent as in Fig. 1.2.

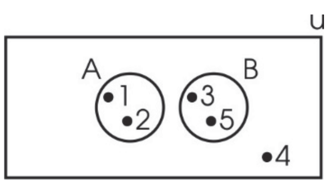


Fig. 1.2

Illustration 12: Let U = {1, 2, 3, 4, 5}, A = {1, 2, 3} and B = {1, 2}. Represent the relationship between U, A and B using Venn diagrams.

Solution: Set U = {1, 2, 3, 4, 5} is the universal set of which A = {1, 2, 3} and B = {1, 2} are subsets. Moreover, in this case we also observe further that B $\subseteq$ A.

Pictorially, we represent this as given in Fig. 1.3.

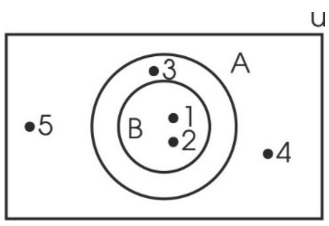


Fig. 1.3

We will further see the extensive use of Venn diagrams when we discuss operations on sets, which are union, intersection and difference of sets.

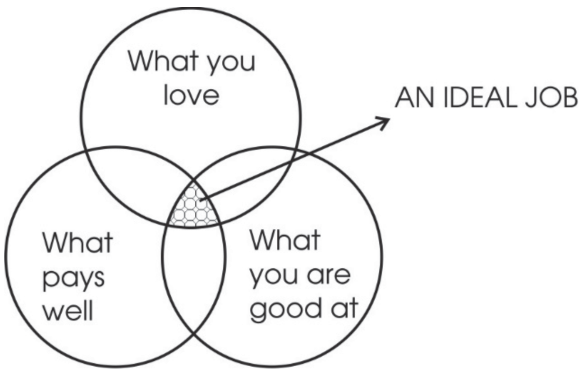
3.10 Operations on Sets

In previous classes we have applied the basic operation of addition, subtraction, multiplication and division on real numbers. Each of these operations on an pair of numbers gives us another number.

For example, when we perform the operation of multiplication on two numbers say 8 and 11, we get a number 88. Now, when we apply the operation of addition on these numbers 8 and 11, we get a number 19.

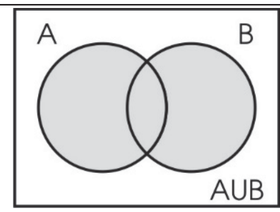
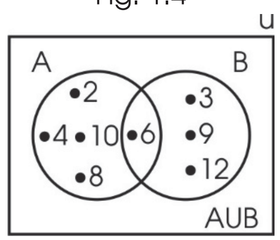
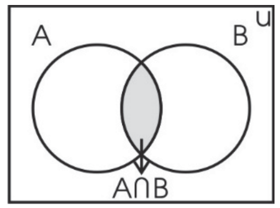
Similarly, there are some operations which when performed on two or more given sets gives rise to another set.

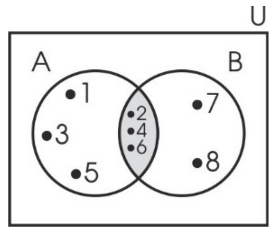
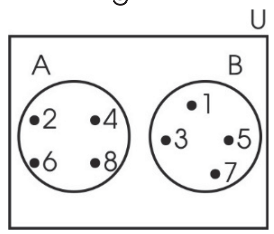
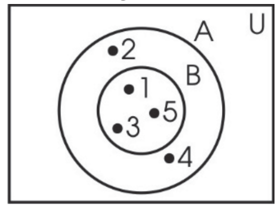
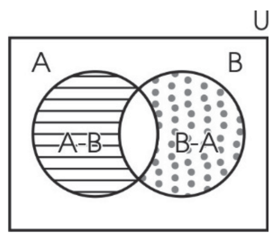
Life's most complicated question can be answered through Venn diagrams and that is choosing an ideal job. A simple Venn diagram can simplify this thought process to a great extent. But, first we need to select the factors which matter in choosing the above, such as, doing what you love, doing something that pays you well and doing what you are good at so, a job which meets all these three criteria, would be a dream job for anyone. Whatever the priority, because you already have listed down the criteria, making the decision becomes easier.

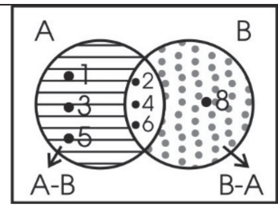
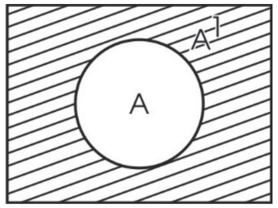
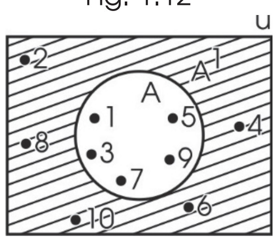


Let us now understand these operations, examine their properties, express them pictorially as Venn diagrams and briefing them through examples.

Table 1.1.2

S. No.	Operations	Symbol	Read as	Definition	Examples	Venn Diagram The shaded portions indicated represent operations	Properties
1.	Union of sets	$A \cup B$ (Fig. 1.4)	A union B	The union of two sets A and B is the set which consists of all those elements which are either in A or in B, including those which are in both, taken once. That is, to say, $A \cup B = \{x : x \in A \text{ or } x \in B\}$	**1.** Let A = {2, 4, 6, 8, 10} and B = {3, 6, 9, 12}. Then $A \cup B$ = {2, 3, 4, 6, 8, 9, 10, 12} (see Fig. 1.5) **2.** Let A = {1, 2, 3, 4, 5}, B = {1, 2, 3}. Then, $A \cup B$ = {1, 2, 3, 4, 5} **Note:** If $B \subseteq A$, then, $A \cup B = A$	 Fig. 1.4  Fig. 1.5	(i) $A \cup B = B \cup A$ (Commutative law) (ii) $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law) (iii) $A \cup \phi = A$ (Law of identity element, ϕ is the identity of U) (iv) $A \cup A = A$ (identity law) (v) $U \cup A = U$ (law of U)
2.	Intersection of sets	$A \cap B$ (Fig. 1.6)	Intersection of sets A and B	The intersection of two sets A and B is the set of all those elements which belong to both A and B. That is, to say, $A \cap B = \{x : x \in A \text{ and } x \in B\}$ **Note:** If A and B are two sets such that $A \cap B = \phi$, that is, there are no elements	**3.** Let A = {1, 2, 3, 4, 5, 6} and B = {2, 4, 6, 7, 8}. Then $A \cap B$ = {2, 4, 6}. (See Fig. 1.7) **4.** Let A = {2, 4, 6, 8} and B = {1, 3, 5, 7}. Then $A \cap B = \phi$. (See Fig. 1.8) **5.** Let A = {1, 2, 3, 4, 5}	 Fig. 1.6	(i) $A \cap B = B \cap A$ (Commutative law) (ii) $\phi \cap A = \phi$ and $U \cap A = A$ (Law of ϕ and U) (iii) $A \cap A = A$ (identity law) (iv) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative law) (v) $A \cap (B \cap C) = (A \cap B) \cup (A \cap C)$ (Distributive law

				which are common to A and B, then A and B are called **disjoint sets**.	and B = {1, 3, 5}. Then A∩B = B. (See Fig. 1.9) **Note:** If B⊆A, then A∩B = B.	 Fig. 1.7  Fig. 1.8  Fig. 1.9	which states that '∩' distributes over '∪')
3.	Difference of sets	A – B (Fig. 1.10)	A minus B	The difference of sets A and B in this order is the set of elements which belong to A but not to B. That is, A – B = {x : x ∈ A and x ∉B}.	**6.** Let A = {1, 2, 3, 4, 5, 6} and B = {2, 4, 6, 8}. Then, A – B = {1, 3, 5} since the elements 1, 3 and 5 belong to A but not B. B – A = {8} because '8' belongs to B and not to A. (See Fig. 1.11)	 Fig. 1.10	(i) A – B ≠ B – A (ii) (A–B) ∪ (A∩B) ∪ (B–A) = A∪B (iii) A – B, B – A and A∩B are disjoint sets.

						 Fig. 1.11	
4.	Complement of a set	A′ (Fig.1.12)	Complement of A.	Let U be the universal set and A be the subset of U. Then the complement of A is the set of all elements of U which are not the elements of A with respect to U. That is, to say, $A' = \{x : x \in U \text{ and } x \notin A\}$.	**7.** Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9 ,10} and A = {1, 3, 5, 7, 9}. Then, A′= {2, 4, 6, 8, 10} (See Fig. 1.13)	 Fig. 1.12  Fig. 1.13	(i) $(A')' = A$ (Law of double complement) (ii) $A \cup A' = U$ $A \cap A' = \phi$ (Complement laws) (iii) De Morgan's law: $(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$ (iv) Laws of empty set and universal set: $\phi' = U$ and $U' = \phi$

Check your progress 3.3

Q.1 For the following sets, find their union and intersection.

(i) A = {x : x is the letter of the word 'MATHEMATICS'}.

B = {x : x is the letter of the word 'TRIGONOMETRY'}.

(ii) A = {x : x is a natural number less than 6}.

B = {x : x is a multiple of 2 from 1 to 10.}

(iii) $A = \{x : x = 2n - 1, n \in N\}$ and $B = \{x : x = 2n, n \in N\}$

(iv) A = {2, 4, 6, 8, 10} and B = {2, 4}

(v) $A = \{x : x = \sin\theta \text{ where } 0 \leq \theta \leq \pi/2\}$

$B = \{x : x = \cos\theta \text{ where } 0 \leq \theta \leq \pi/2\}$

Q.2 Let U = {1, 2, 3, 4, 5, 6, 7, 8}; A = {1, 2, 3, 4}; B = {3, 4, 6}; C = {5, 6, 7, 8}, find:

(i) $A - (B \cup C)$

(ii) $A \cap C'$

(iii) $B' \cap C'$

(iv) $B' \cup A'$

(v) $A - (B \cup C)'$

Q.3 Let U = {1, 2, 3, 4, 5, 6, 7, 8}; A = {1, 2, 3}; B = {2, 4, 6}, C = {3, 6, 9, 12}. Verify.

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

(iii) $A' \cap (B \cup C) = (A' \cap B) \cup (A' \cap C)$

(iv) $A' \cup B = (A \cap B')'$

(v) $A' - B' = B - A$

Q.4 Draw suitable Venn diagrams for each of the following.

(i) $(A \cup B)'$

(ii) $(A \cap B)'$

(iii) $A' \cap B'$

(iv) $A' \cup B'$

Q.5 Two finite sets have 'm' and 'n' elements. The total number of subsets of the first set is 56 more than the total number of subsets of the second set. Find the values of 'm' and 'n'.

Q.6 Let A = {a, b, c} and B = {a, b, c, d}. Is $A \subseteq B$? What is $A \cup B$? What is $A \cap B$?

Q.7 Fill in the blanks.

(i) $A' \cap \phi =$ ___________

(ii) $A \cap \phi' =$ ___________

(iii) $A \cup U' =$ ___________

(iv) $A \cap U' =$ ___________

(v) $A \cap \phi' =$ ___________

(vi) $U' \cap \phi =$ ___________

(vii) $U' \cap \phi' =$ ___________

(viii) $U' \cup \phi =$ ___________

(ix) $U' \cup \phi' =$ ___________

(x) $U \cup \phi' =$ ___________

Q.8 If A = {1, 2, 3, 4}, then the number of subsets of set A containing element 3, is:

(i) 24

(ii) 28

(iii) 8

(iv) 16

Q.9 If A = {1, 2, 3, 4, 5}, B = {2, 4, 6} and C = {3, 4, 6}, then $(A \cup B) \cap C$ is

(i) {3, 4, 6}

(ii) {1, 2, 3}

(iii) {1, 4, 3}

(iv) None of these

3.11 Practical Problems on Operations on Sets

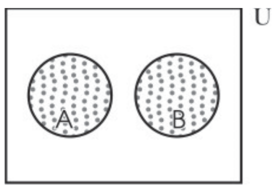
In this section, we will now apply the concept of sets to our daily life situations and solve some problems.

Let A, B and C be finite sets.

(i) $n(A \cup B) = n(A) + n(B)$; if $A \cap B = \phi$ i.e. A and B are disjoint.

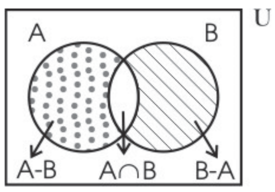
(ii) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ in general

Now, case (i) follows immediately as the elements in $A \cup B$ are either in A or in B but not in both because $A \cap B = \phi$.



Let's move ahead with case (ii)

Now, $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$, where, $A - B$, $B - A$ and $A \cap B$ are mutually disjoint.



Therefore, $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$

$$= n(A - B) + n(A \cap B) + n(B - A) + n(A \cap B) - n(A \cap B)$$

$$= n(A) + n(B) - n(A \cap B)$$

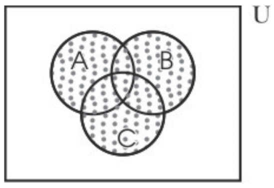
(iii) If A, B and C are finite sets, then,

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

Now, proceeding with the proof of how we obtain the formula for case (iii).

$n(A \cup B \cup C) = n(A \cup (B \cup C))$

$= n(A) + n(B \cup C) - n(A \cap (B \cup C))$ (from case ii)



$= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap (B \cup C))$

Since, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$\therefore$ $n(A \cap (B \cup C)) = n(A \cap B) \cup (A \cap C))$

$= n(A \cap B) + n(A \cap C) - n((A \cap B) \cap (A \cap C))$

$= n(A \cap B) + n(A \cap C) - n(A \cap B \cap C)$

$\therefore$ $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap C) - n(A \cap B) + n(A \cap B \cap C)$

Illustration 13: Out of 20 members in a family, 12 like tea and 15 like coffee. Assume that each one likes atleast one of the two drinks, how many like.

(i) Both coffee and tea.

(ii) Only tea and not coffee

(iii) Only coffee and not tea

Solution:

(i) Let T denote the set of people who like tea and C be the set of people who like coffee.

Therefore, $n(T) = 12$, $n(C) = 15$ and $n(T \cup C) = 20$

Using the formula, $n(T \cup C) = n(T) + n(C) - n(T \cap C)$

$20 = 12 + 15 - n(T \cap C)$

$\therefore$ $n(T \cap C) = 12 + 15 - 20 = 7$

Hence 7 people like both tea and coffee.

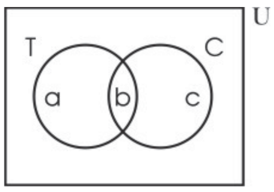
(ii) $n(T \cap C') = n(T) - n(T \cap C) = 12 - 7 = 5$

Therefore, 5 people like to take tea but not coffee.

(iii) Now, $n(C \cap T') = n(C) - n(T \cap C) = 15 - 7 = 8$

Therefore, 8 people like to take coffee and not tea.

Alternatively: Using Venn diagram, we can understand the problem as:



(a) Let a be the number of people who like tea but not coffee.

(b) Let b be the number of people who like both tea and coffee.

(c) Let c be the number of people who like coffee but not tea.

According to the question,

$a + b + c = 20$ (i)

$a + b = 12$ (ii)

$b + c = 15$ (iii)

Form (i) and (ii), $c = 8$.

Substituting $c = 8$ in (iii), we get $b = 7$ and substituting $b = 7$ in (ii) we get $a = 5$. Therefore,

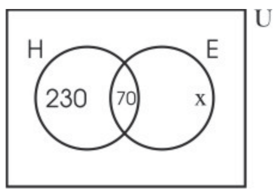
(i) $n(C \cap T) = b = 7$

(ii) $n(T \cap C') = a = 5$

(iii) $n(C \cap T') = c = 8$

Illustration 14: In a group of 400 people, who speak either Hindi or English or both, if 230 speak Hindi only and 70 speak both English and Hindi, then how many of them speak English only?

Solution:



According to the question, $230 + 70 + x = 400$

$\Rightarrow \quad x = 400 - 300 = 100$

$\Rightarrow \quad x = 100$

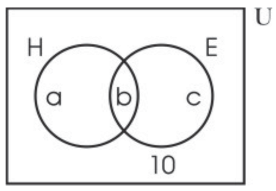
$\therefore$ The number of people who speak English only is 100.

Illustration 15: In a group of 70 people, 45 speak Hindi language and 33 speak English language and 10 speak neither Hindi nor English. How many can speak both English as well as Hindi language? How many can speak only English language?

Let H = the set of people who speak Hindi

E = the set of people who speak English

Solution: According to the question, out of 70 people, 10 speak neither Hindi nor English. Therefore, $n(H \cup E) = 70 - 10 = 60$.



Let a be the number of people who speak Hindi but not English, b be the number of people who speak both Hindi and English and c be the number of people who speak only English.

Now, according to the question,

$a + b + c = 60 \qquad$ (i)

$a + b = 45 \qquad$ (ii)

$b + c = 33 \qquad$ (iii)

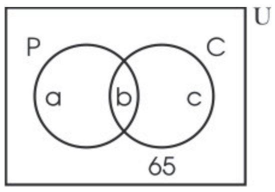
Therefore, from (i) and (ii), the number of people speaking English language only is c = 15. Now, substituting c = 15 in (iii) we get number of people speaking English and Hindi is b = 18.

Illustration 16: In a certain town, 25% of the families own a phone. 15% own a car and, 65% families own neither a phone nor a car. 2000 families own both a car and a phone. Find how much percentage of families own either a car or a phone. Also, find how many families live in the town.

Let P= the set of families who own a phone

C= the set of families who own a car

Solution: Let the number of families in the town be 100.



According to the questions,

$a + b = 25$ (i)

$b + c = 15$ (ii)

$a + b + c = 100 - 65 = 35$ (iii)

Solving, (i), (ii) and (iii), we get,

$a = 20$, $b = 5$ and $c = 10$

$\therefore$ Percentage of families, owning either a car or a phone is 35% as $a + b + c = 20 + 5 + 10 = 35$

Now, let 'x' be the number of families living in the town.

Therefore, $\frac{5}{100} \times x = 2000$ (given)

$\therefore$ $x = 40{,}000$

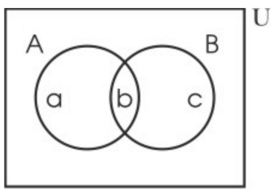
$\therefore$ The number of families in the town are 40,000

Illustration 17: Out of 500 car owners investigated, 400 owned car A and 200 owned car B, 50 owned both A and B cars. Is the data correct?

Let A= the set of Car owners of Car A

B= the set of Car owners of Car B

Solution:



$n(U) = 500$

$a + b = 400$

$b + c = 200$

$b = 50$

$\Rightarrow$ $a = 350$ and $c = 150$

$\therefore$ $a + b + c = 350 + 150 + 50 = 550 \neq 500$

Now, according to the above solution and given question,

$n(A \cup B) > n(U)$, which is not possible as $n(AUB) \neq n(U)$

Therefore, the above data is incorrect.

Illustration 18: Let U be the universal set for sets A and B such that $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$.

Then, $n(A' \cap B') = 300$, provided $n(u)$ is equal to:

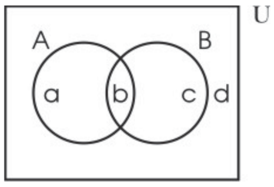
(i) 600

(ii) 700

(c) 800

(d) 900

Solution:



According to the question,

$a + b = 200$ (i)

$b + c = 300$ (ii)

$b = 100$

$d = 300$

Adding (i) and (ii),

$a + b + b + c = 500$

$\Rightarrow$ $(a + b + c) + b = 500$

$\Rightarrow$ $a + b + c = 500 - b = 500 - 100 = 400$

$\therefore$ $n(A \cup B) = 400$

Now, $n(U) = n(A \cup B) + n((A \cup B)')$

$= n(A \cup B) + n(A' \cap B')$

$= 400 + 300$

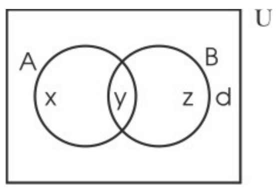
$= 700$

Illustration 19: A market research group conducted a survey of 1000 consumers and reported that 720 consumers like product A and 450 consumers like product B. What is the least number that must have liked both the products?

Let A = the set of consumers who like product A

B = the set of consumers who like product B

Solution:



n (U) = 1000, let n (A) = x + y, n (b) = y + z and n ($A \cap B$) = y

$\therefore$ $x + y = 720$ (i)

$y + z = 450$ (ii)

Now, $n(A \cup B) \leq n(U)$

$\therefore$ $x + y + z \leq 1000$ (iii)

Adding (i) and (ii) we get

$(x + y + z) + y = 1170$

Now from (iii), as $x + y + z \leq 1000$

$\therefore$ $(x + y + z) + y \leq 1000 + y$ (adding y on both sides)

$\therefore$ $1170 \leq 1000 + y$

$\therefore$ $y \geq 170$

Therefore, the least number that liked both the products is 170.

Illustration 20: In a survey of 60 people, it was found that 25 people read newspaper H; 26 read newspaper T, 26 read newspapers I, 9 read both H and I, 11 read both H and T, 8 read both T and I and, 3 read all three newspapers. Find:

(i) the number of people who read atleast one of the newspapers.

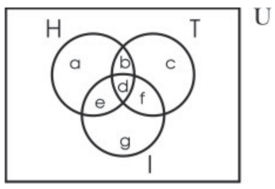
(ii) the number of people who read exactly one newspaper.

Let H = the set of people who read newspaper H

T = the set of people who read newspaper T

I = the set of people who read newspaper I

Solution:



According to the question, n (u) = 60

$a + b + d + e = 25$ (i)

$b + c + d + f = 26$ (ii)

$e + d + f + g = 26$ (iii)

$e + d = 9$ (iv)

$b + d = 11$ (v)

$d + f = 8$ (vi)

$d = 3$ (vii)

Substituting (vii) in (iv), (v) and (vi), we get,

$f = 5, b = 8, e = 6$

From, (i), (ii) and (iii), $g = 12$, $c = 10$ and $a = 8$

(i) the number of people reading atleast one newspaper is

$a + b + c + d + e + f + g = 8 + 8 + 10 + 3 + 6 + 5 + 12 = 52$

(ii) the number of people who read exactly one newspaper are

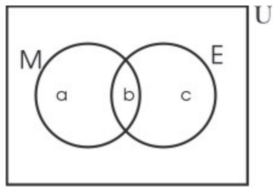
$a + c + g = 8 + 10 + 12 = 30$

Illustration 21: In a class of 35 students, 17 have taken Mathematics, 10 have taken Mathematics but not Economics. If each student has taken either of the two subjects, then find the number of students who have taken Economics but not Mathematics.

Let M = the set of students who have taken Maths

E = the set of students who have taken Economics

Solution:



According to the question,

$a + b + c = 35$ (i)

$a + b = 17$ (ii)

$a = 10$

We need to find the number of students who have taken Economics but not Mathematics i.e. we need to find 'c' Subtracting (ii) from (i), we get c = 18.

Illustration 22: With corona virus threating to run riot in India, prevention appears to be the best cure available so far. It is crucial for people to have awareness and knowledge about the virus and need to take proper precautions to discourage its spread. A survey was conducted on 25 people to see if proper precautions were being taken by people and following points were observed:

(a) 15 people used face masks

(b) 14 consciously maintained social distancing

(c) 5 used face masks and washed their hand regularly

(d) 9 maintained social distancing and used face masks.

(e) 3 were practicing all the three measures.

(f) 4 maintained social distancing and washing hands regularly.

(g) 4 practised only social distancing norms.

Assuming that everyone took at least one of the precautionary measures, find:

(i) How many exercised only washing hands as precautionary measure.

(ii) How many pratised social distancing and washing hands but not wearing masks.

(iii) How many exercised only wearing masks.

Solution:

Consider the Venn diagram as in Fig. 1.14. Let S denote the set of people maintaining social distancing M denote the set of people wearing fall masks and W denote the people washing hands regularly.

Let a, b, c, d, e, f, g denote the number of people in the respective regions.

n (maintaining social distancing) = n (S) = a + b + d + e = 14 (i)

n (wearing face mask = n (M) = d + e + f + g = 15 (ii)

n (washed hands and wear face masks) = n $(W \cap M)$ = e + f = 5 (iii)

n (who took all the these measures) = n $(S \cap W \cap M)$ = e = 3 (iv)

n (maintaining distancing & washing hands) = n $(S \cap W)$ = b + e = 4 (v)

n (maintained only social distancing = n $(S \cap W' \cap M')$ = a = 4 (vi)

n (maintained social distancing and wear face masks

$= n(S \cap M) = e + d = 9$ (vii)

Now, a = 4, e = 3. Showing the above questions:

$n(S \cap W) = b + e = 4 \Rightarrow b = 1$

$n(S \cap M) = e + d = 9 \Rightarrow d = 6$

$n(W \cap M) = e + f = 5 \Rightarrow f = 2$

$n(M) = d + e + f + g = 15 \Rightarrow g = 4$

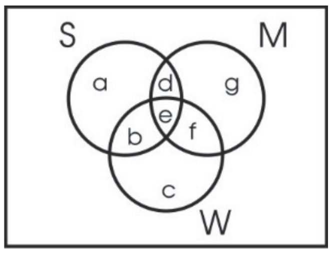
Now, a + b + c + d + e + f + g = 25

$\Rightarrow$ c = 5

$\therefore$ (i) Number of people who exercised only washing hands = c = 5

(ii) Number of people who practised social distancing and washing hands but not wearing masks = b = 1

(iii) Number of people who exercised only wearing masks = g = 4



Check your progress 3.4

Q.1 In a group of 70 people, 37 like coffee, 52 like tea and each person likes atleast one of the two drinks. How many people like both coffee and tea.

Q.2 In a group of 65 people, 40 like cricket, 10 like both cricket and tennis. How many like tennis only? How many like tennis.

Q.3 In a survey of 600 students in a school, 150 students were found to the taking apple juice, 225 taking orange juice and 100 were taking both apple and orange juice. Find how many were taking neither apple juice nor orange juice.

Q.4 In an election, two contestants A and B contested. y% of the total votes voted for A and (y + 30)% for B. If 20% of the voters did not vote, then y =

(i) 30

(ii) 25

(iii) 40

(iv) 35

Q.5 In a class, 70 students wrote two tests, test-I and test-II. 50% of the students failed in test-I and 40% of the students in test-II. How many students passed in both the tests?

(i) 21

(ii) 7

(iii) 28

(iv) 14

Solution: Check Your Progress 3.1

1. (ii), (iii), (iv) are sets.

2. (i) A = {x : x = 4n, where n = 1, 2, 3, 4, 5}

   (ii) B = {x : x is a prime number}

   (iii) C = {x : x is a consonant}

   (iv) D = {x : x is a solution of the equation $x^2 - 1 = 0$}

   (v) E = {x : x is a prime number between 40 and 50}

   (vi) F={x:x=$\frac{1}{n}$, where n $\in$ N

   (vii) G = {x : x = $\frac{1}{n^2}$ where n $\in$ N

3. (i) A = {1, 2, 3, 4}

   (ii) B = {–3, –2, –1, 0, 1, 2, 3, 4, 5, 6}

   (iii) C = {F, O, L, W}

   (iv) D = {15, 24, 33, 42, 51, 60}

   (v) E = {A, R, I, T, H, M, E, C}

4. (i) c

   (ii) e

   (iii) a

   (iv) b

   (v) d

5. (i) $\in$

   (ii) $\notin$

   (iii) $\in$

   (iv) $\in$

(v) $\not\in$

Solution: Check Your Progress 3.2

1. (i) Finite; cordiality : 99

   (ii) Infinite

   (iii) Finite; cordiality : 7

   (iv) Infinite

   (v) Infinite

   (vi) Infinite

2. (i) Singleton

   (ii) Empty set

   (iii) Singleton

   (iv) Singleton

   (v) Empty

3. (i), (ii), (iii)

4. (i) $\in$

   (ii) $\subset$

   (iii) $\not\in$

   (iv) $\subset$

   (v) $\not\in$

   (vi) $\subset$

   (vii) $\subset$

5. (i) $\{x : x \in R, 1 < x < 3\}$

   (ii) $\{x : x \in R, -1 \leq x \leq 3\}$

(iii) $\{x : x \in R, -4 < x \leq 0\}$

(iv) $\{x : x \in R, -1 \leq x < 1\}$

(v) $\{x : x \in R, 0 \leq x < \infty\}$

6. $P(P(A)) = 2$

7. 16

Solution: Check Your Progress 3.3

1. (i) $A \cap B = \{M, T, E, I\}$; $A \cup B = \{M, A, T, H, E, I, C, S, R, G, O, N, Y\}$

   (ii) $A \cap B = \{2, 4\}$; $A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}$

   (iii) $A \cap B = \phi$; $A \cup B = N$

   (iv) $A \cap B = \{2, 4\}$; $A \cup B = \{2, 4, 6, 8, 10\}$

   (v) $A \cap B = \{0, 1\}$; $A \cup B = \{0, 1\}$

2. (i) $A - (B \cup C) = \{1, 2\}$

   (ii) $A \cap C) = \{1, 2, 3, 4\}$

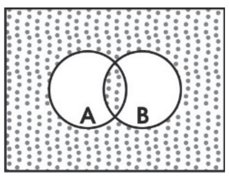
   (iii) $B' \cap C' = \{1, 2\}$

   (iv) $B' \cup A' = \{1, 2, 5, 6, 7, 8\}$

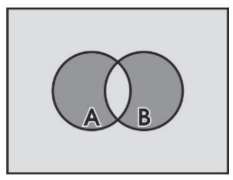
   (v) $A - (B \cup C)' = \{3, 4\}$

4. (i) $(A \cup B)'$: (ii) $(A \cap B)'$:

(iii) $A' \cap B'$:

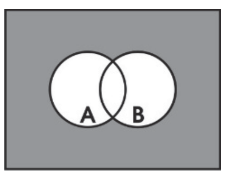


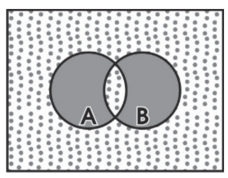
(iv) $A' \cup B'$



:

5. m = 6, n = 3





6. Yes; B; A

7.	(i)	ϕ	(ii)	A
	(iii)	A	(iv)	ϕ
	(v)	A	(vi)	ϕ
	(vii)	ϕ	(viii)	ϕ
	(ix)	U	(x)	U
8.	(iii)	8		
9.	(i)	{3, 4, 6}		

Solution: Check Your Progress 1.4

1. 19
2. 25; 35
3. 325
4. (iii) 25
5. (ii) 7

Summary

This chapter deals with basic definitions, operations and practical problems involving sets. These are summarised below:

- A set is a well-defined collection of distinct objects.
- Representation of sets in Roster or Tabular form and set builder form.
- Types of sets:

  (a) A set which does not contain any element is called an empty set.

  (b) A set which contains only one element is called a singleton set.

  (c) A set which contains definite number of elements is called a finite set, otherwise it is called an infinite set.

  (d) Two sets are said to be equal if they have exactly the same elements.

  (e) A set 'A' is said to be a subset of a set 'B', if every element of A is also an element of B.

  (f) Power set of a set A is the set of all the subsets of A and is denoted by P(A).

  (g) A set that contains all the elements in a given context is called a universal set and is denoted by U.

  (h) Intervals are also subsets of R, set of real numbers.

- Venn diagrams
- Operations on set

  (a) Union of two sets A and B is the set of all those elements which are either in A or in B.

  (b) Intersection of two sets A and B is the set of all elements which are common to both A and B.

(c) Difference of sets A and B in this order is the set of elements which belong to A but not to B.

(d) The complement of a set A, which is a subset of U, the universal set, is the set of all elements of U which are not the elements of A.

- De Morgan's Law: For any two sets A & B, $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$.

- If A and B are finite sets, then if:

(a) $A \cap B = \phi$, then $n(A \cup B) = n(A) + n(B)$

(b) $A \cap B \neq \phi$, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Chapter IV

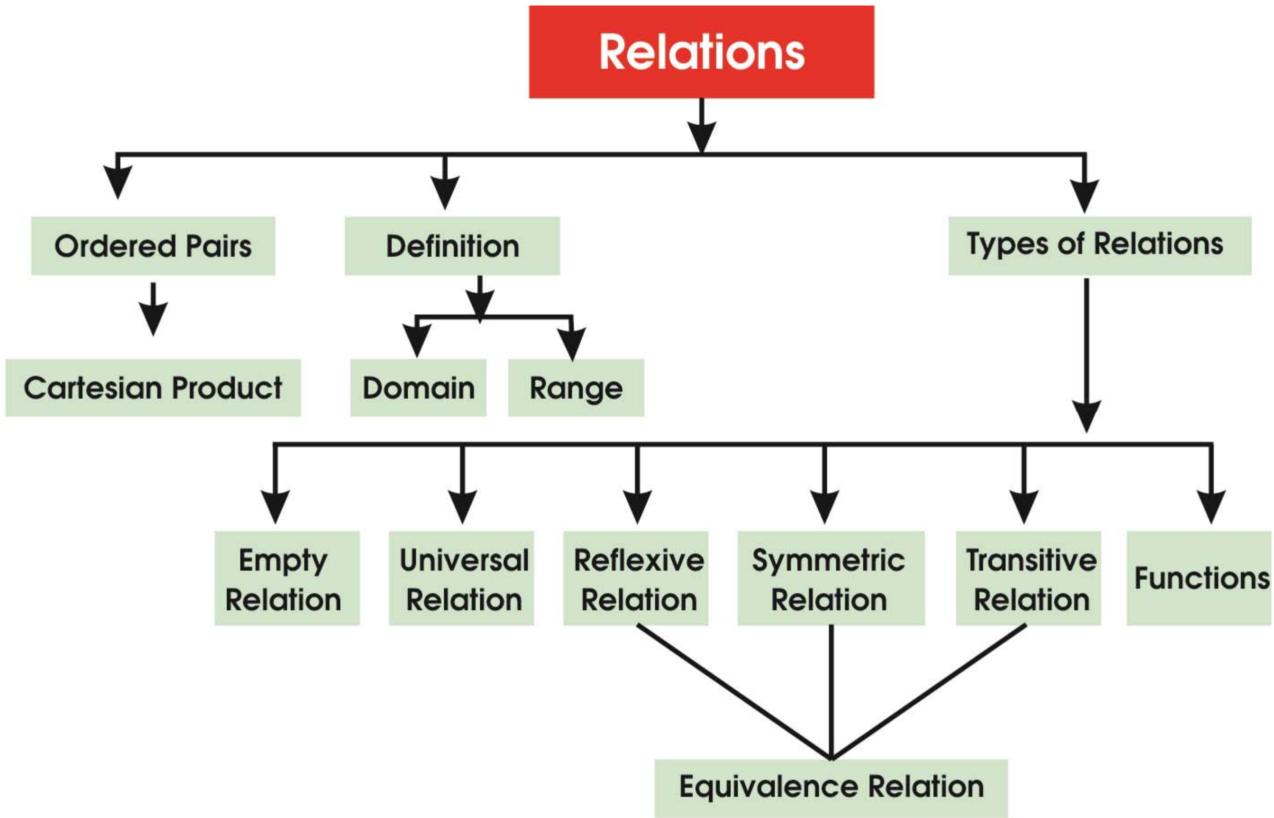
Relations

Learning Objectives:

After the completion of this section, the students will be able to:

- Define an ordered pair.
- Understand the concept of cartesian product of sets.
- Relate relations as a subset of cartesian product.
- Identify different types of relations.

Concept Map



4.1 Introduction

A relation is a way in which two or more people or things are connected. In day to day life, we come across many such relations, such as, brother and sister, mother and daughter, teacher and student, table and its cost price and the list is endless. In mathematics too, we come across many relations, for e.g., number 'y' is twice the number 'x', set A is a subset of set B, a line l_1 is parallel to line l_2, volume of a sphere with its radius r is $\frac{4}{3}\pi r^3$ and so on. In all these examples we observe that a relation involves pairs of objects in a certain order.

In this chapter, we will understand how to connect pairs of objects from two sets and then define a relation between two objects of the pair. After having understood the concept of relations, we would do different types of relations.

For understanding 'relation', we first need to know about ordered pairs and Cartesian product of sets.

4.2 Ordered Pair

What is an ordered pair?

Definition 1: An ordered pair is a pair of objects taken is a specific order and the order in which they appear in the pair is significant. The ordered pair is written as (a, b), where 'a' is the first member of the pair and 'b' is the second member of the pair.

For example, if we form pairs of stationery items along with its price, we may write it as (pencil, Rs. 1), (pen, Rs. 10), (eraser, Rs. 5) and so on. It is evident from these examples, that the first member represents the stationery item and second member, its cost per piece.

Definition 2: Equality of ordered pairs:

Two ordered pairs (a, b) and (c, d) are said to be equal if a = c and b = d, that is, to say,

$$(a, b) = (c, d) \Leftrightarrow a = c \text{ and } b = d$$

Note:

1. The order in which the elements of the pair occur should be meaningful.
2. The ordered pairs (a, b) and (b, a) are different, unless a = b.
3. Neither (a, b) represents an interval nor does it represent a set {a, b}.

Illustration 1: If the ordered pairs $(2x, y - 3)$ and $(2, 1)$ are equal, then find x and y.

Solution: Since, $(2x, y - 3) = (2, 1)$, we have

$$2x = 2 \text{ and } y - 3 = 1$$

$$\Rightarrow \quad x = 1 \text{ and } y = 4$$

Illustration 2: Find 'x' and 'y' if $(x^2 - 4x, y^2 - y) = (-4, 6)$

Solution: Since, $(x^2 - 4x, y^2 - y) = (-4, 6)$

$\Rightarrow$ $x^2 - 4x = -4$ and $y^2 - y = 6$

$\Rightarrow$ $x^2 - 4x + 4 = 0$ and $y^2 - y - 6 = 0$

$\Rightarrow$ $(x - 2)^2 = 0$ and $(y - 3)(y + 2) = 0$

$\Rightarrow$ $x = 2$ and $y = -2, 3$

4.3 Cartesian Product of Two Sets

Let P and Q be two non-empty sets. Then the Cartesian product of P and Q in this order is written as P x Q and is defined as the set of all ordered pairs (p,q) such that $p \in P$, $q \in Q$ that is,

$$P \times Q = \{(p, q) : p \in P, q \in Q\}$$

Illustration 3: Let A = {a, b, c}, B = {1, 2}

Find A x B and B x A. Is A x B = B x A?

Solution: Given A = {a, b, c} and B = {1, 2}, then by definition of Cartesian product, we have,

$$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$

$$B \times A = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}$$

Here, A x B and B x A do not have exactly the same elements (ordered pairs) $\therefore$ $A \times B \neq B \times A$

We conclude that, in general, $A \times B \neq B \times A$.

In case A x B = B x A, then A = B

Note:

1. In general $A \times B \neq B \times A$. In case A x B = B x A, then A = B.
2. $A \times B = \varnothing$ if either A or B or both are empty sets.
3. $A \times B \neq \varnothing$ if both A and B are non-empty sets.
4. n (A x B) = n(A) x n(B), where n (A) denotes the number of elements in the set A, that is,

if n (A) = p and n (B) = q, then

n (A x B) = pq

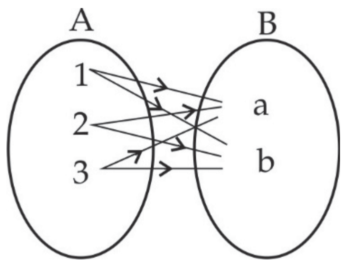
5. A x B x C = {(a, b, c) : a $\in$ A, b $\in$ B, c $\in$ C}

   Here, (a, b, c) is called an ordered triplet and n(A x B x C) = n(A) x n(B) x n(C)

6. If A and B are non-empty sets and either A or B is infinite then (A x B) is also an infinite set.

Let us understand Cartesian product of two sets, say, A and B, through arrow diagrams.

Let A = {1, 2, 3} and B = {a, b}. Then n(A x B) = 3 x 2 = 6



A x B = {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}

Illustration 4: Let A = {2, 3} and B = {4, 5}. Find:

1. A x B
2. B x A
3. n (A x B)
4. number of subsets of A x B

Solution: Now, A = {2, 3}, B = {4, 5}

1. A x B = {(2, 4), (2, 5), (3, 4), (3, 5)}
2. B x A = {(4, 2), (4, 3), (5, 2), (5, 3)}
3. n (A x B) = n (A) x n(B) = 2 x 2 = 4
4. Now n(A x B) = 4

∴ Number of subsets of A x B are $2^4 = 16$

Note: A x B and B x A are equivalent sets as the n(A x B) = n(B x A)

Illustration 5: If A x B = {(a, x), (a, y), (b, x), (b, y)}, then find A and B.

Solution: A x B = {(a, x), (a, y), (b, x), (b, y)}

Then, A = {a, b} and B = {x, y}

Illustration 6: If (–1, 1), (2, 3), (1, 0), (2, 1) are some of the elements of A x B, then find A and B. Also find the remaining elements of A x B.

Solution: Given (–1, 1), (2, 3), (1, 0), (2, 1) ∈ A B

⇒ A = {–1, 1, 2}, B = {0, 1, 3}

∴ n (A x B) = 9

Therefore, there are 5 remaining elements which are:

(–1, 3), (–1, 0), (2, 0), (1, 1) and (1, 3)

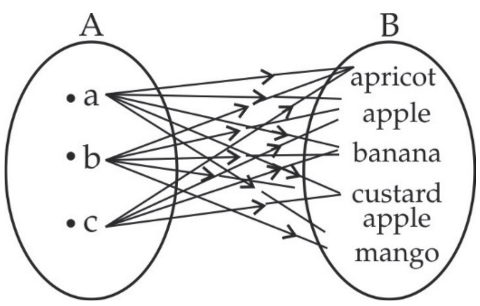
4.4 Relations

Consider two non-empty sets, say, A = {a, b, c} and set B = {apple, apricot, banana, custard apple, mango}.

Now, the Cartesian product of A and B, i.e., A x B has 15 elements, listed as

A x B = {(a, apple), (a, apricot), (a, banana)..... (c, mango)}

Refer to Fig. 1 to depict A x B using arrow diagram



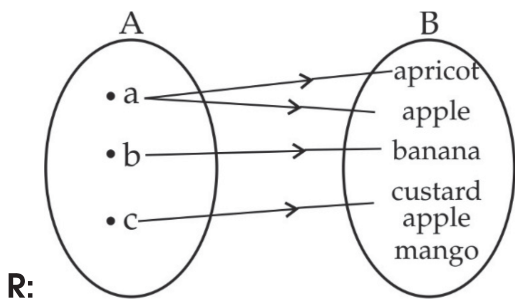
We can now obtain a subset of A x B by defining a relation R between the first element belonging to A and second belonging to B of each ordered pair (a, b) as:

R = {(a, b) : a is the first letter of the name of the fruit in B, a $\in$ A, b $\in$ B}

Therefore, R = {(a, apple), (a, apricot), (b, banana), (c, custard apple)}

Here, R is said to be a 'Relation from the set A to the set B'

An arrow diagram of this relation R is as shown in Figure 2.



Note: A Relation 'R' from a set A to a set B is a subset of A x B.

Definition 3: A relation 'R' from a non-empty set A to a non-empty set B is defined to be a subset of Cartesian product A x B. If (a,b) $\in$ R, then the second elements is called the image of the first element a. Further, we say that a is R - related to b and we write a R b.

Definition 4: The set of all the first elements of the ordered pairs in a relation 'R' from set A to a set B is called the domain of the relation R.

Definition 5: The set of all second elements in a relation R from a set A to a set B is called the range of the relation R. The entire set B is called the co-domain of the relation R.

Note: Range $\subseteq$ co-domain

Illustration 6: Let A = {1, 2, 3, 4, 5, 6}. Define a relation R from A to A by:

R = {(x, y) : x > y}

Note : A Relation from A to itself is called a relation defined on A (or simply, a relation in A)

1. Write the given relation in tabular or roster form.

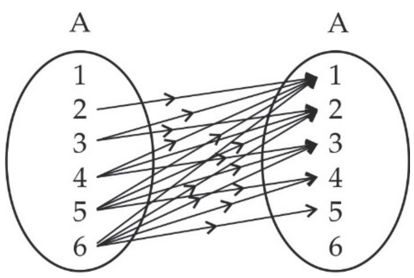
2. Define this relation using an arrow diagram.

3. Write the domain, range and co-domain of R.

Solution:

1. R = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3), (5, 4), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)}

2.



3. Domain = {2, 3, 4, 5, 6}

   Range = {1, 2, 3, 4, 5}

Co-domain = A={1,2,3,4,5,6}

Illustration 7: Let A be a non-empty set with n(A) = 3. Find the number of relations from set A to itself.

Solution: By the definition, a relation from any set A to set B is a subset of A x B.

Now, n(A x A) = n(A) n(A) = 9

$\therefore$ Number of subsets of A x A are 2^9.

Note: Let R be a relation from set A to set B, where n(A) = p and n(B) = q.

Then the total number of relations from A to B are given by 2^{pq} as relations are subsets of A x B and number of subsets of the set A x B containing pq elements are 2^{pq}.

4.5 Types of Relations

In this section we would be discussing different types of relations. In this section, we will also be dealing with functions from set A to itself.

Definition 6: Empty Relation

A relation R in a set A is called empty relation, if no element of A is related to any element of A, i.e., R = $\phi \subset$ A x A.

Illustration 8: Let A be the set of all natural numbers. Define R = {(x, y); $\frac{x}{y} < 0$; x, y $\in$ A}.

Solution: Since all natural numbers are always greater than zero, therefore, division of no two natural numbers will give a negative real number. This shows that relation R on A x A is an empty relation.

Definition 7: Universal Relation

A relation R in a set A is called universal relation, if each element of A is related to every element of A, i.e., R = A x A.

Illustration 9: Let A = {1, 2, 3, 4, 5}

Define R = {(x, y) : x + y $\in$ N; x, y $\in$ A}

Solution: R = {(x, y) : x + y $\in$ N; x, y $\in$ A}

= {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5)}

= A x A

Hence, R is a universal relation

Equivalence relations

Let us introduce equivalence relation, which is one of the most important relations and plays a very significant role. In order to study equivalence relation we first understand the three types of relations, i.e., reflexive, symmetric and transitive.

Definition: A relation R in a set A is called

1. Reflexive, if $(a, a) \in R$, for every $a \in A$
2. Symmetric, if $(a, b) \in R \Rightarrow (b, a) \in R$, for all $a, b \in A$
3. Transitive, if $(a, b) \in R$ and $(b, c) \in R$

   $\Rightarrow (a, c) \in R$, for all $a, b, c \in A$

Definition: A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

Example 1: Let T be the set of all triangles in a plane with R a relation in T given by

$R = \{(T_1, T_2) : (T_1 \sim T_2\}$. Show that R is an equivalence relation.

Solution: *R is reflexive Let* $T_1 \in T$

Since every triangle is similar to itself.

i.e, $T_1 \sim T_1 \;\; \forall T_1 \in T$

This means that $(T_1, T_1) \in T \;\underline{\text{V}}\; T_1 \in T$

Therefore, R is reflexive relation

R is symmetric Let $T_1, T_2 \in T$ such that $T_1 \sim T_2$

Then $T_2 \sim T_1$ we conclude R is a symmetric relation.

R is transitive Let $T_1, T_2, T_3 \in T$ such that $(T_1, T_2), (T_2, T_3) \in T$

Then $T_1 \sim T_2$ and $T_2 \sim T_3 \Rightarrow T_1 \sim T_3 \Rightarrow (T_1, T_3) \in T$

Hence R is also transitive

Since R is reflexive, symmetric and transitive. Therefore R is an equivalence relation.

Example 2: Let S be any non empty set and R be a relation defined on power set of S i.e. on P(S) by A R B iff $A \subset B$ for all A, B $\in$ P(S). Show that R is reflexive and transitive but not symmetric.

Solution: R is reflexive

For every A $\in$ P(S), $A \subset A$ i.e., A R A

R is not symmetric

We have $\varnothing$, A $\in$ P (s) where A contains at least one element. $\subset \varnothing$ A, But A $\varnothing \not\subset$

Hence R is not symmetric

R is transitive

For A, B, C $\in$ P(S), if $A \subset B$ and $B \subset C \Rightarrow A \subset C$

Thus for, (A, B), (B, C) $\in R \Rightarrow$ (A, C) $\in$ R $\forall$ A, B, C $\in$ P(S)

Hence R is transitive

Example 3: Show that the relation R in the set Z of integers given by

$$R = \{(a, b) : 3 \text{ divides } a - b\}$$

is an equivalence relation.

Solution: *R is reflexive*

As 3 divides (a – a) for every $a \in Z$, $\therefore$ $(a, a) \in R$ $\forall a \in Z$

R is symmetric

If $(a, b) \in R$ then 3 divides a – b

i.e. $a - b = 3\lambda$, where λ is any integer

$\Rightarrow b - a = -3\lambda$

then 3 divides b – a

$\Rightarrow (b, a) \in R$

Therefore R is symmetric

R is transitive

If $(a, b), (b, c) \in R$

Then 3 divides (a – b) and (b – c)

$a - b = 3\lambda$ and $b - c = 3\mu$, where λ and μ are intergers

$a - c = (a - b) + (b - c) = 3\lambda + 3\mu$

$a - c = 3(\lambda+\mu)$

$a - c = 3k \Rightarrow a - c = 3$ (an Integer)

3 divides $a - c \Rightarrow (a, c) \in R$

This shows R is transitive

Thus, R is an equivalence relation in Z.

Exercise

1. Determine whether each of the following relations are reflexive, symmetric and transitive.
   (i) Relation R in a set S = {1, 2, 3, 4, 5} as R = {(x, y) : y is divisible by x}
   (ii) Relation R in a set L of all lines in a plane as $R = \{(L_1, L_2) : L_1 \perp L_2\}$
2. Show that the relation R in the set R of real numbers, defined as $R = \{(a, b) : a < b^2\}$ is neither reflexive nor symmetric nor transitive.
3. Show that the relation R in the set Z of integers given by R = {(a, b) : 2 divides a – b} is an equivalence relation.
4. If X = {1, 2, 3, 4}, give an example on X which is
   (i) reflexive and symmetric but not transitive.
   (ii) Symmetric and transitive but not reflexive.
   (iii) Neither reflexive, nor symmetric but transitive.
5. If R_1 and R_2 are equivalence relations in set A, show that $R_1 \cap R_2$ is also an equivalence relation.

Solution:

1. (i) reflexive, not symmetric, transitive
   (ii) not reflexive, symmetric, not transitive

Functions

After having understood the concept of relations, let's now briefly try to understand special type of relations called functions, which will later be discussed in detail in calculus.

Definition: Let A and B be two non-empty sets. A function 'f' from set A to set B is a special type of relation from A to B which assigns to every element in A a unique image in B. Set A, is called the domain of 'f' and elements of set B which have pre-image in A is called the range of 'f'. Set B is called co-domain of 'f'.

Therefore, Range $\subseteq$ co-domain

The function 'f' from A to B is denoted by f: A $\rightarrow$ B. Consider the following example.

Example: Examine each of the following relations given below and state, giving reasons whether it is a function or not?

(i) R = {(2, 1), (3, 2), (4, 1), (5, 2), (3, 3)}

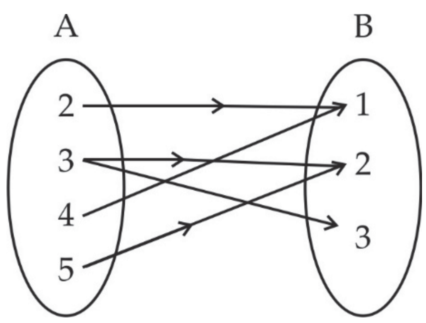
(ii) R = {(1, 2), (2, 3), (3, 4), (4, 5)}

(iii) R = {(1, 1), (2, 1), (3, 1), (4, 1)}

(iv) R = {(1, 1), (1, 2), (1, 3)}

Solution: In this case, domain of R, say, A = {2, 3, 4, 5} and range, say, B = {1, 2, 3}.

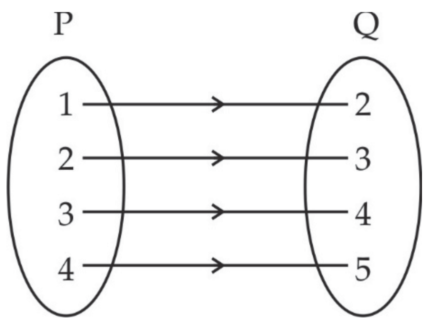
Consider the following arrow diagram from set A to set B.



The given relation is not a function from A to B as there exists an element '3' in the domain which does not have a unique image. (It has two images 2 and 3)

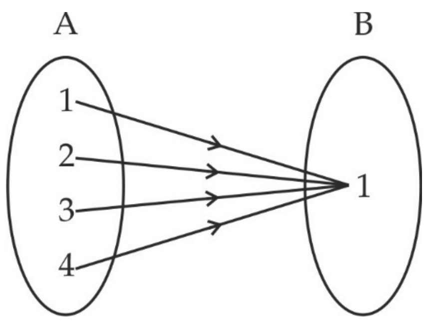
(ii) In this case, domain of R, say, P = {1, 2, 3, 4} and range, say, Q = {2, 3, 4, 5}

According to the arrow diagram, depicting this relation, is given below:



R is a function as every element in P has a unique image in Q.

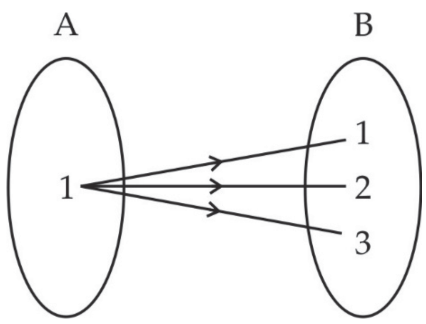
(iii) Consider set A={1,2,3,4} and set B={1} Representing the relation by arrow diagram



Relation R is a function from A to B as every element of set A have unique image in set B. Domain of function is A = {1,2,3,4} and range B = {1}

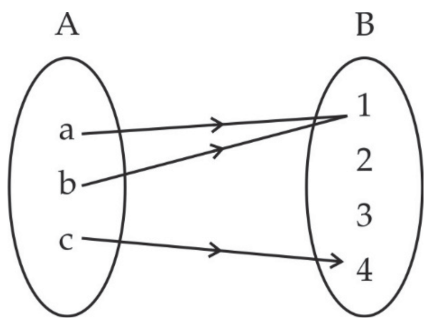
Consider case

(iv). Here, domain of the relation R, is, say A = {1} and range, say B = {1, 2, 3}. Representing this relation pictorially, we get:

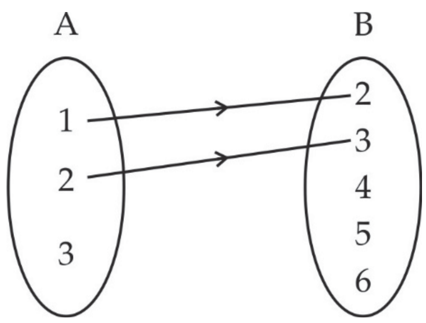


This relation R is not a function from A to B as the element '1' in the domain doesn't have a unique image in B.

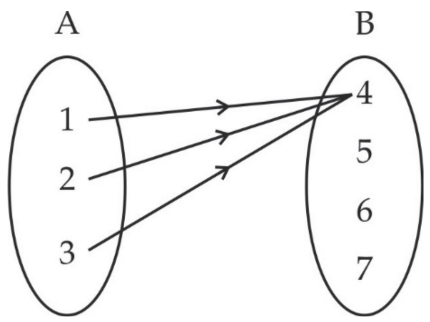
Example: Consider the following arrow diagrams depicting relations from set A to set B. Which amongst them are functions? Give reasons.



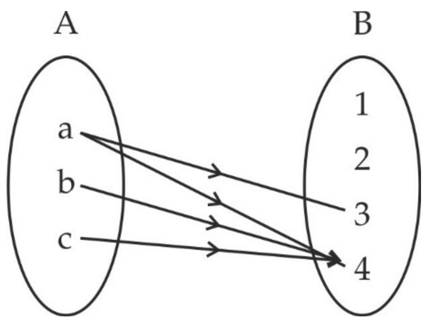
(i)



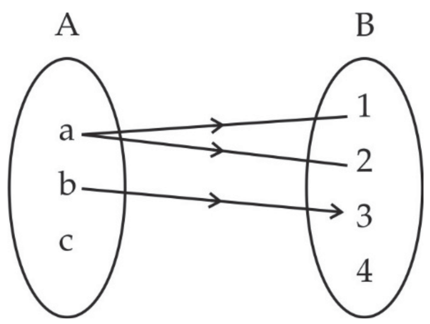
(ii)



(iii)



(iv)



(v)

Solution: Only Fig. (i) and Fig. (iii) depict functions as every element in set A has a unique image in set B.

In Fig. (ii), the relation is not a function as '3' doesn't have an image in B. Hence, not a function.

In Fig. (iv) element 'a' has two images in set B. Hence, not a function.

In Fig. (v), element 'a' has two images in B and also element 'c' has no image in 'B'. Hence, not a function.

Example: Find the domain and range of each of the following real functions:

(i) $f(x) = x^2$

(ii) $f(x) = \frac{1}{x}$

Solution:

(i) $f(x) = x^2$

Domain: R, set of real numbers

Range: $(0, \infty)$ or non-negative real numbers.

(ii) $f(x) = \frac{1}{x}$

Domain: $R - \{0\}$, Range: $R - \{0\}$

Summary :

The main features of this section are :

- Ordered pair : A pair of object or elements in a specific order
- Cartesian product of A x B of two sets A and B is given by

  A x B ={(a,b) : a $\in$ A and b $\in$ B.
- If n(a) = p and n(B) = q then n (A x B) = pq and number of relations possible from A to B are 2^{pq}
- In general, A x B $\neq$ B x A. In case A x B = B x A, then A = B.
- Relation 'R' from a non-empty set A to a non-empty set B is a subset of Cartesian product A x B, which is derived by defining a relationship between the first element and the second element of the ordered pair A x B.
- Image is the second element of the ordered pairs in the given relation
- The domain of a relation R is the set of all the first elements of ordered pairs in a relation R.
- The range of a relation R is the set of all second elements of ordered pairs in relation R.
- A Relation R in a set A is called an empty relation if no element of A is related to any element of A.
- A relation R in a set A is called a universal relation if each element of A is related to every element of A
- A relation R on a set A is said to be:

  i) Reflexive if (a,a) $\in$ R, for every a $\in$ A

  ii) Symmetric if (a,b) $\in$ R $\Rightarrow$ (b,a) $\in$ R, where a, b $\in$ A

  iii) Transitive if (a,b) $\in$ R and (b,c) $\in$ R then (a,c) $\in$ R where a,b,c $\in$ A
- A relation R which is reflexive, symmetric and transitive is called an equivalence relation.

Chapter - V

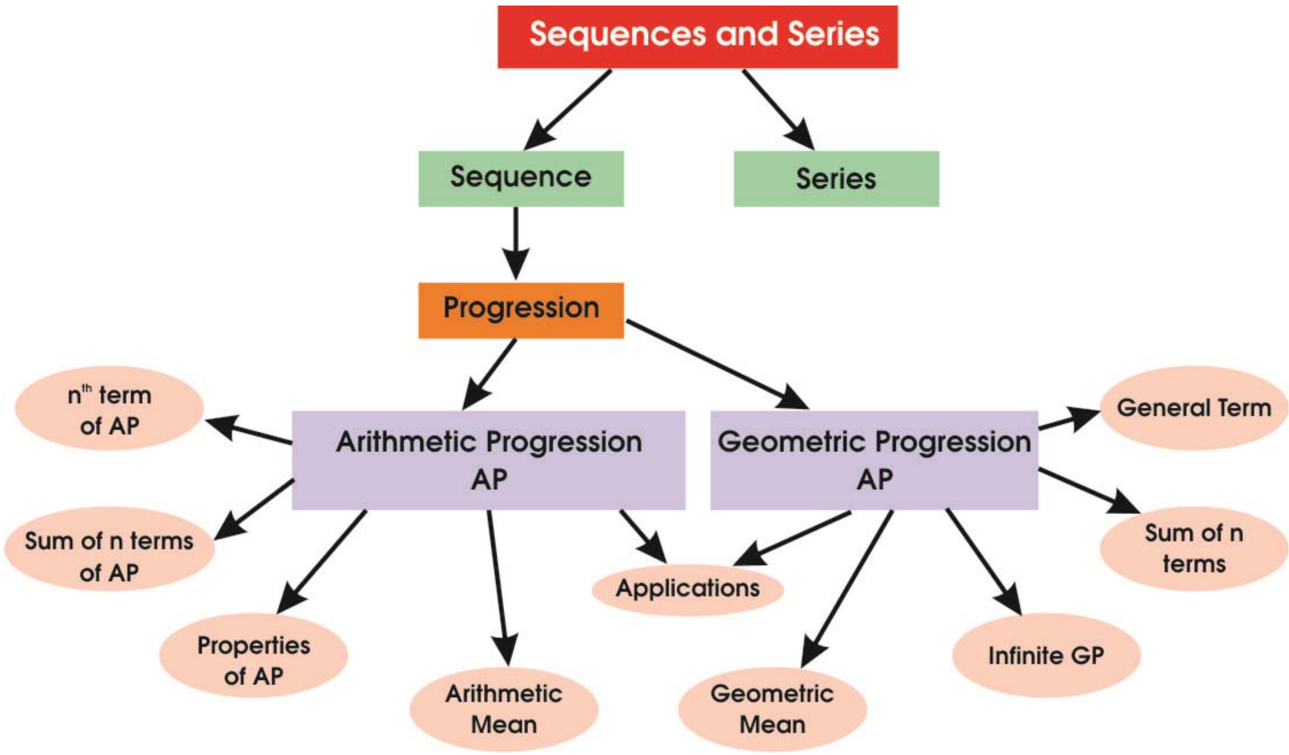
Sequences and Series

Learning Objectives:

After completion of this unit you will be able to:

- Differentiate among sequence, series and progressions
- Identity different type of progressions i.e. AP, GP
- Explain the concepts of AP and GP
- Derive the various formulas used in the chapter
- Solve problems based on different concepts
- Apply the knowledge gained to solve daily life problems

Concept Map :



5.1 Introduction

As in ordinary English, in mathematics too, sequence means a particular order in which one thing follow another.

This in turn means as in that ordered collection, we would be able to identify first, second, third members and so on. For example, the house numbers when you walk down the street, the population growth whether it be in human beings or bacteria, the amount of money deposited in bank, all form a sequence; hence conforming that sequences do have important applications in many spheres of life.

Sequence, following a specific pattern is called progression. In standard tenth, you have studied about arithmetic progression or A.P. In this chapter in addition to A.P., we would be discussing about arithmetic mean, geometric progression, geometric mean and relationship between arithmetic mean and geometric mean.

5.2 Sequences

Let us consider the following example:

A TV manufacturer has produced 1200 TVs in his first year of production. Assuming the production increases uniformly by a fixed number every year, say 200 per year. Find the number of TV sets produced in 5th year.

Now here, starting from the first year, the number of TV sets produced in 1st, 2nd, 3rd, 4th & 5th year would be 1200, 1400, 1600, 1800 and 2000 respectively. These numbers form a sequence.

Moreover, this type of sequence which has countable number of terms is called a finite sequences, otherwise it is an infinite sequence, where the number of elements is uncountable.

The various numbers occurring in the sequence are called terms and are denoted by a_1, a_2, ... a_n, where the subscripts denote the position of the term in the squence. The n^{th} term is the number at the n^{th} position of the sequence and is denoted by a_n.

This n^{th} term is called the general term of the sequence. So, in the above example,

$a_1 = 1200, a_2 = 1400, ... a_5 = 2000$

A sequence may not only be obtained by adding a fixed constant to the preceding term as seen in the above example, but it may also be obtained by multiplying a fixed non-zero real number to the preceding term.

For example, a child has 2 parents, further those parents have two parents, and so continuing in this manner, the child has 4 grandparents, 8 great grandparents, and so on. Now, in this case the sequence formed may be noted as 2, 4, 8, ... and so on.
This type of sequence is obtained by multiplying the preceding term by non-zero constant 2. This type of sequence is called a geometric progression. Here, $a_1 = 2$, $a_2 = 4$ and so on and the multiplying factor 2 is called the common ratio.

Series

Let a_1, a_2, ... a_n be a given sequence. Then the expression $a_1 + a_2 + a_3 + ... + a_n + ...$ is called the series associated with a given sequence.

5.3 Arithmetic Progression

5.3.1 Definition 1: A sequence a_1, a_2, a_3, ..., a_n, ... is called an arithmetic progression (A.P.), if the difference of any term and its immediate preceding term is always the same, i.e.

$$a_{n+1} - a_n = \text{constant, say, 'd', for all } n \varepsilon N,$$

or $\quad a_{n+1} = a_n + d. \qquad \forall\ n\varepsilon N$

This constant difference 'd' is called the common difference. The term 'a_1' in the above sequence is called the first term.

Let us consider all the terms of the sequence a_1, a_2, ... a_n, ... in terms of only the first term and the common difference.

Let the first term 'a_1' be denoted by 'a' and common difference by 'd'. Therefore,

$$a_2 = a_1 + d = a + d = a + (2 - 1)d$$

$$a_3 = a_2 + d = (a + d) + d = a + 2d = a + (3 - 1)d$$

$$a_4 = a_3 + d = (a + 2d) + d = a + 3d = a + (4 - 1)d$$

$$a_n = a + (n - 1)d$$

Hence, the n^{th} term (general term) of the AP is:

$$a_n = a + (n - 1)d$$

Illustration 1: Find the first three terms of the arithmetic progressions whose n^{th} term is given. Also find the common difference and the 20th term in each case.

(i) $a_n = 2n + 5$

Now $a = a_1 = 2(1) + 5 = 7$

$a_2 = 2(2) + 5 = 9$

$a_3 = 2(3) + 5 = 11$

Therefore, $d = a_2 - a_1 = 9 - 7 = 2$

$\therefore$ a_{20} can either be found from 'a_n'=2n+5 putting n=20 or given by substituting the value of 'a' & 'd' in an.

$\therefore$ either; $a_{20} = 2(20) + 5 = 45$

or $a_{20} = a + (n - 1)\, d = 7 + (20 - 1) \times 2$

$= 7 + 2 \times 19 = 45$

(ii) $a_n = 3 - 4n$

$a_1 = 3 - 4\,(1) = -1$

$a_2 = 3 - 4\,(2) = -5$

$a_3 = 3 - 4\,(3) = -9$

Therefore, $d = a_2 - a_1 = -5 - (-) = -4$

You, may also calculate d as $d = a_3 - a_2 = -9 - (-5) = -4$

Hence, $a_{20} = a + (n - 1)d = -1 + 19\,(-4) = -77$

(iii) $a_n = \frac{n-3}{4}$

$a_1 = \frac{1-3}{4} = -\frac{1}{2}$

$a_2 = \frac{2-3}{4} = -\frac{1}{4}$

$a_3 = \frac{3-3}{4} = 0$

Therefore, $d = a_2 - a_1 = -\frac{1}{4} - \left(-\frac{1}{2}\right) = \frac{-1}{4} + \frac{1}{2} = \frac{1}{4}$

or $\quad d = a_3 - a_2 = 0 - \left(-\frac{1}{4}\right) = \frac{1}{4}$

Here $a_{20} = \frac{20-3}{4} = \frac{17}{4}$

Hence, from above Illustration we may note that the terms of an AP or the common difference, both can be any real number.

Let's Check on Some Properties of an A.P.

5.3.2 Properties of an A.P.

Property (i): If a constant is added to each term of an A.P., the resulting sequence is also an A.P. with same common difference as the previous one.

Example: Let us consider an AP: 1, 3, 5, 7, ...

In this sequence, $d = 2$

Now add a constant, say, 5, to each term,

We get a new sequence, 6, 8, 10, 12, ... which is also an A.P. with common difference $d = 2$.

Note that though the terms in the two sequences change, but the common difference remains the same.

Property (ii): If a constant is subtracted from each term of an A.P., the resulting sequence is also an A.P., again, with same common difference as the previous sequence.

Example: Let us consider an A.P.: 8, 5, 2, –1, –4, ...

In this sequence, $d = -3$

Now subtract a constant, say, 5, from each term.

We get a new sequence: 3, 0, –3, –6, –9, ..., which is again A.P., an with the same common difference d = –3.

Property (iii): Similarly if each term of an AP is multiplied by a constant, then the resulting sequence is also an A.P.

Example: Let us consider the sequence: 5, 9, 13, 17, ..., an A.P.

Here, d = 4

Now, multiply each term by, say, –4, we get a new sequence.

–20, –36, –52, –68, ... which is again an A.P., with d = –16

Please note in this case the common difference is –4d.

So, we may conclude that the common difference in this case is the product of the common difference and the number with which each term is multiplied.

So, if in an A.P., with common difference, 'd', each term is multiplied with a non-zero constant, say ' b', then the common difference of new AP would be 'bd'.

Property (iv): If each term of an A.P. is divided by a non-zero constant, then, the resulting sequence is also an A.P. In this case, the common difference of the new A.P. will be obtained by dividing the common difference of the original sequence by the non-zero constant with which each term was divided.

Property (v): A sequence is an A.P. if its n^{th} term is a unique linear expression in n i.e., a_n = An + B. In such cases, the coefficient of 'n' i.e. A is always the common difference.

Property (vi): In a finite A.P., a_1, a_2, a_3, ... a_n the sum of the terms equidistant from the beginning and end is always the same and is equal to the sum of its first and last term.

i.e. if $a_1, a_2, a_3, \ldots\ldots, a_{n-2}, a_{n-1}, a_n$ is an A.P. then

$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots\ldots$

Property (vii): If the terms of an A.P. are chosen at regular intervals, then they form an A.P.

i.e. if $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ are in AP, then

a_1, a_3, a_5, a_7, chosen at same interval are also in A.P.

Property (viii): Three numbers a, b, c are in A.P. if 2b = a + b

This is because, $b - a = c - b \Rightarrow 2b = a + c$

Let us understand, what are the applications of arithmetic progression in daily life.

(a) **The Saving Schemes:** When you save some amount of money in the bank, you earn an increment to it. For example, if you deposit Rs. 1000 and every year you earn an increment of Rs. 100 this would make an A.P.. You can also calculate the total amount received by you in a particular year that you require.

(b) **A Taxi Fare:** A taxi ride can cost you fixed change. For example, for the first one kilometer, it can cost you Rs. 25 and the next subsequent kilometer that you cover, it would cost you additional Rs. 10. For each kilometer. Therefore, Rs. 25, Rs. 35, Rs. 45, ... would make an A.P.

Illustration 2: In a sequence, the n^{th} term is $a_n = 2n^2 + 5$. Show that it is not an A.P.

Solution: Given: $a_n = 2n^2 + 5$

$a_1 = 7, a_2 = 13, a_3 = 23, a_4 = 37\ldots\ldots$

Now, $a_2 - a_1 = 6$

$a_3 - a_2 = 10$

Since the common difference between any two consecutive terms is not the same, therefore it is not an A.P.

Illustration 3: Find the number of identical terms in the two sequences:

1, 5, 9, 13, 17, ..., 197 and 1, 4, 7, 10, 13, ... 196

Solution: We will first find the number of terms in the two sequences.

Let the number of terms in first sequence be 'n'.

Therefore, 197 = a + (n - 1)d, as 197 is the last n^{th} term of the sequence.

$\therefore \quad 197 = 1 + (n-1) \times 4 = 1 + 4(n-1)$

$\Rightarrow \dfrac{196}{4} = n - 1 \Rightarrow n = 50$

Let the number of terms in the second sequence be 'm'.

This implies, 196 = a + (m-1)d

$\therefore \quad 196 = 1 + (m-1) \times 3 = 1+3(m-1)$

$\Rightarrow \dfrac{195}{3} = m - 1$

$\Rightarrow m = 66$

Now, According to the question, let p^{th} term of first sequence be equal to the q^{th} term of the second sequence.

$\therefore \quad a_1 + (p-1)\, d_1 = A_1 + (q-1)\, D_1$

$1 + 4(p-1) = 1 + 3(q-1)$

$\therefore \quad \dfrac{p-1}{3} = \dfrac{q-1}{4} = k$ (say)

$\therefore \quad p = 3k + 1,\ q = 4k + 1$

Now since first A.P. contains 50 terms, therefore,

$p \leq 50 \Rightarrow 3k + 1 \leq 50 \Rightarrow K \leq \dfrac{49}{3}$

$\Rightarrow k \leq 16\frac{1}{3}$

Since second A.P. contains 66 terms, therefore,

$$q \le 66 \Rightarrow 4k + 1 \le 66$$

$$\Rightarrow k \le \frac{65}{4} = 16\frac{1}{4}$$

But since k is a natural number, therefore, $k \le 16$

$\therefore$ k = 1, 2, 3,..... 16

Hence, there are 16 identical terms in the given sequence.

Illustration 4: Find the 10^{th} term from the end of the sequence 7, 10, 13, ..., 130

Solution: The given sequence is an A.P. with common difference d = 3 and first term a = 7

Now, since 130 is the last term of the sequence, therefore,

$$a_n = 130$$

$\Rightarrow$ $a + (n - 1)\,d = 130$

$\Rightarrow$ $7 + (n - 1) \times 3 = 130$

$\Rightarrow$ $123 = 3\,(n - 1)$

$\Rightarrow$ $n = 42$

Now, 10^{th} term from the end of the sequence 7, 10, 13, ... 13, would be the 10^{th} term from the beginning of the sequence 130, ..., 13, 10, 7, obtained by reversing the order.

In this case, a = 130, d = –3, n = 42

$\therefore$ $a_{10} = a + (n - 1)\,d$

$$= 130 + (10 - 1)(-3)$$

$$= 130 - 27 = 103$$

Illustration 5: If p times the p^{th} term of an A.P. is q times then q^{th} term, the show that its $(p + q)^{th}$ term is zero.

Solution: Given p $a_p = q\ a_q$

To prove: $a_{p+q} = 0$

Now, let 'a' be the first term and 'd' be the common difference of the sequence.

$\therefore \quad a_p = a + (p - 1)d$

$\quad a_q = a + (q - 1)d$

According to the question,

$\quad p(a_p) = q(a_q)$

$\Rightarrow \quad p\{a + (p - 1)d\} = q\{a + (q-1)d\}$

$\Rightarrow \quad ap + p(p - 1)d = aq + q(q - 1)d$

$\Rightarrow \quad a(p - q) + d(p^2 - p - q^2 + q) = 0$

$\Rightarrow \quad a\ (p - q) + d\ ((p - q\ (p + q) - (p - q)) = 0$

$\Rightarrow \quad a(p - q) + d\ ((p - q)\ (p + q - 1)) = 0$

$\Rightarrow \quad (p - q)\ (a + d(p + q - 1)) = 0$

$\Rightarrow \quad a + (p + q - 1)d = 0 \qquad (\text{as } p - q \neq 0 \text{ as } p \neq q)$

$\Rightarrow \quad a_{p+q} = 0$

Hence Proved.

Illustration 6: If the numbers a, b, c, d, e form an A.P., then the value of a – 4b + 6c – 4d + e is:

(1)	1	(b)	2
(c)	0	(d)	non of these

Solution: Let D be the common difference of the A.P. Then,

$\quad b = a+D,\ C = a+2D,\ d = 3D,\ e = a+4D$

So $\quad a - 4b + 6c - 4d + e$

$= a - 4(a + D) + 6(a + 2D) - 4(a + 3D) + (a + 4D)$

$= a - 4a - 4D + 6a + 12D - 4a - 12D + a + 4D$

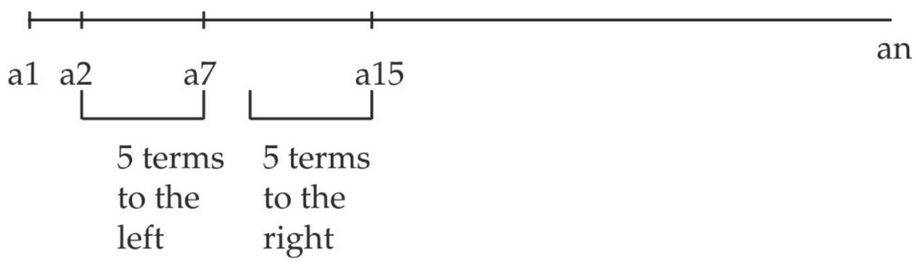
$= 0$

Illustration 7: If a_n be the n^{th} term of an A.P. and if $a_7 = 15$, then the value of the common difference that would make the product $a_2\, a_7\, a_{12}$ greatest is:

(1) 9 (b) 9/4

(c) 0 (d) 18

Solution: Let d be the common difference of the A.P. Then,



$a_2\, a_7\, a_{12} = (15 - 5d)\, 15\, (15 + 5d) = 375\, (9 - d^2) \qquad (1)$

Therefore, if $a_7 = 15$, $a_2 = 15 - 5d$, $a_{15} = 15 + 5d$

Hence, RHS is greatest when $d = 0$ (as $d^2 \geq 0$)

Therefore, $a_2\, a_7\, a_{12}$ is greatest for $d = 0$

5.3.3 Selection of Terms of an A.P.

Sometimes, we require only certain number of terms of an A.P. The following ways of selecting terms are often convenient to find the solution of the problem.

Table 5.3.1

Number of Terms	Terms in an A.P.	Common Difference
3	a – d, a, a + d	d
4	a – 3d, a – d, a + d, a + 3d	2d
5	a – 2d, a – d, a, a + d, a + 2d	d
6	a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d	2d

Please note that in case of odd number of terms, the middle term is 'a' and common difference is d while in case of an even number of terms the middle terms are a – d and a + d and the common difference is 2d.

5.3.4 Sum of First 'n' Terms of an A.P.

Consider the terms of the A.P. $a_1, a_2, a_3, \ldots, a_n$, with common difference d.

Now, $a_1 = a$ (first term)

$a_2 = a + d$

$a_3 = a + 2d$

$a_n = a + (n - 1)d$

Adding these terms together we get,

$a_1 + a_2 + a_3 \ldots + a_n = a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d)$

$= na + (1 + 2 + \ldots + (n - 1))d$

$= na + \frac{n(n-1)}{2}d$

$= \frac{n}{2}\left[2a+(n-1)d\right]$

∴ Sum of first 'n' terms of an A.P., denoted by Sn, is given by

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

This can further be re-written as:

$$Sn = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$= \frac{n}{2}\left[a+a+(n-1)d\right]$$

$= \frac{n}{2}\left[a+\ell\right]$, where '$\ell$' denotes the last term a_n

$$\therefore Sn = \frac{n}{2}\left[a+\ell\right]$$

Note: If Sn denotes the sum of 'n' terms, then $S_2 - S_1 = a_2$ as $S_1 = a_1$ (first term) and S_2 is the sum of first two terms a_1 and a_2.

Hence, $S_1 = a_1$; $S_2 = a_1 + a_2$

$$\Rightarrow S_2 - S_1 = a_2$$

If we generalise, $S_n - S_{n-1} = a_n$

Illustration 8: Find the sum of first 100 even natural number.

Solution: Consider the sequence 2, 4, 6, 8,..... of even natural numbers.

Now the first term is a = 2, first term and the common difference is d = 2

$\therefore \quad a_{100} = a + (n-1)d$

$= 2 + 99 \times 2$

$= 200$

Now, $S_{100} = \frac{100}{2}\ (2 \times 2 + 99 \times 2)$

$= 50 \times (4 + 198)$

$= 50 \times 202$

$= 10100$

After $a_{100} = 200$

As $\quad S_n = \frac{n}{2}[a+\ell]$

$\therefore S_{100} = \frac{100}{2}(2 + 200)$

$= 50 \times 202 = 10100$

Illustration 9: How many terms of the A.P. 17, 15, 13, are need to give the sum 72. Explain the double answer.

Solution: Let the number of terms to give the sum 72 be 'n'.

$\therefore \quad S_n = \frac{n}{2}(2 \times 17 + (n - 1) \times (-2))$

Now, according to the question

$S_n = 72$

$\therefore \quad 72 = \frac{n}{2}(34 - 2n + 2)$

$= \frac{n}{2}(36 - 2n) = 72$

$= n(18 - n) = 72$

$\therefore \quad n(n - 18) + 72 = 0$

$= n^2 - 18n + 72 = 0$

$\Rightarrow \quad n = 12, 6$

The double answer is due to the fact that some terms in the A.P. are negative and these terms on addition with positive numbers gives zero.

Illustration 10: The sum of three numbers in A.P. is 24 and their product is 440. Find the numbers.

Solution: According to the table, 1.3.1 let the three numbers in A.P. be a – d, a, a + d.

According to the question, a – d + a + a + d = 24.

$$3a = 24 \Rightarrow a = 8$$

Also, (a – d) a (a + d) = 440

$$a(a^2 - d^2) = 440$$

$$8(a^2 - d^2) = 440$$

$$a^2 - d^2 = \frac{440}{8} = 55$$

$$64 - d^2 = 55$$

$$d^2 = 9 \Rightarrow d = \pm 3$$

When a = 8, d = 3, then the numbers are 5, 8, 11

When a = 8, d = –3, then the numbers are 11, 8, 5

Therefore, the required numbers are 5, 8, 11

Illustration 11: Solve for x: 1 + 4 + 7 + 10 + ... + x = 590

Solution: The given series is an arithmetic series, with a = 1 and d = 3.

Now, 'x' in the question represents the last term of the series and let us consider the number of terms to be 'n'. Therefore, $x = a_n$

According to the question, $\frac{n}{2}\left[2a+(n-1)d\right] = 590$

$$\Rightarrow \frac{n}{2}\left[2\times1+(n-1)\times3\right] = 590$$

$$\Rightarrow n(2 + 3n - 3) = 590 \times 2$$

$$\Rightarrow n(3n - 1) = 1180$$

$$\Rightarrow 3n^2 - n - 1180 = 0$$

$\Rightarrow (n - 20)(3n + 59) = 0$

$\Rightarrow n = 20, n = -\frac{59}{3}$

$n = -\frac{59}{3}$ is not possible as 'n' is a natural number.

$\therefore \quad n = 20$

$\therefore \quad x = a_n = a + (n - 1)d = 1 + (20 - 1) \times 3$

$= 1 + 57 = 58$

Illustration 12: If the first, second and last terms of an A.P. are a, b & c respectively, then show that the sum of terms in A.P. is $\frac{(a+c)(b+c-2a)}{2(b-a)}$.

Solution: Let the number of terms in the given A.P. be in 'n' and common difference be 'd'

Therefore, according to the question,

$a_1 = a$

$a_2 = b = a + d \Rightarrow d = b - a$

$a_n = c = a + (n - 1)d$

$\Rightarrow \quad c = a + (n - 1)(b - a)$

$\Rightarrow \quad c - a = (n - 1)(b - a)$

$\Rightarrow \quad \frac{c-a}{b-a} = n - 1$

$\Rightarrow \quad \frac{c-a}{b-a} + 1 = n$

$\Rightarrow \quad n = \frac{b+c-2a}{b-a}$

Now, the sum of all the terms in the given A.P. is $S_n = \frac{n}{2}(a+\ell)$

$$\Rightarrow \quad S_n = \frac{n}{2}[a+c]$$

$$= \frac{(b+c-2a)}{2(b-a)}[a+c]$$

Illustration 13: The sum of n terms of two APs are in the ratio (3n + 8) : (7n + 15). Find the ratio of their 12th terms.

Solution: Let a_1, a_2 and d_1, d_2 be the first terms and common difference of the first and second A.P. respectively. According to the question,

$$\frac{\text{sum of n terms of first A.P.}}{\text{sum of n terms of second A.P.}} = \frac{3n+8}{7n+15}$$

$$\Rightarrow \quad \frac{\frac{n}{2}\left[2a_1+(n-1)d_1\right]}{\frac{n}{2}\left[2a_2+(n-1)d_2\right]} = \frac{3n+8}{7n+15}$$

$$\Rightarrow \quad \frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2} = \frac{3n+8}{7n+15} \qquad (1)$$

Now, we need to find the ratio of 12th terms

$$\therefore \quad \frac{12^{th}\text{ terms of first A.P.}}{12^{th}\text{ terms of second A.P.}} = \frac{a_1+11d_1}{a_2+11d_2}$$

$$= \frac{2a_1+22d_1}{2a_2+22d_2} = \frac{2a_1+(23-1)d_1}{2a_2+(23-1)d_2} \qquad (2)$$

From (1) and (2)

$$\frac{2a_1+(23-1)d_1}{2a_2+(23-1)d_2} = \frac{3\times(23)+8}{7\times(23)+15} \qquad \text{(Put n = 23)}$$

$$\Rightarrow \quad \frac{12^{th}\text{ terms of first A.P.}}{12^{th}\text{ terms of second A.P.}} = \frac{7}{16}$$

5.3.5 Arithmetic Mean

If between any two number say 'a' and 'b', we insert a number, say 'A' such that a, A, b is an A.P., then this number A is called the arithmetic mean of the number 'a' and 'b' i.e.,

$$A - a = b - A$$

$$\Rightarrow \qquad A = \frac{a+b}{2}$$

From this we may interpret that the AM between two numbers 'a' and 'b' is their average $\frac{a+b}{2}$

For example, consider two number 5 and 15. We may observe that 10 = $\frac{15+5}{2}$ = $\frac{a+b}{2}$ is the AM between 5 and 15.

More generally, given two numbers a and b, we need not restrict to only one arithmetic mean between any two given number. We can insert as many AMs between any two numbers as we like such that the resulting sequence is an A.P.

Let $A_1, A_2, A_3, \ldots A_n$ be 'n' numbers between 'a' and 'b' such that a, $A_1, A_2, A_3, \ldots A_n$, b forms an AP. These $A_1, A_2, \ldots, A_n$ are called the AMs of the given sequence. Note that the first AM is the second term of the sequence, second AM is the third term of the sequence and so on.

Illustration 14: Insert 6 numbers between 3 and 24 such that the resulting sequence is an A.P.

Solution: Let A_1, A_2, A_3, A_4, A_5 and A_6 be six numbers between 3 and 24 such that 3, A_1, A_2, A_3, A_4, A_5, A_6, 24 are in A.P.

Here a = 3, n = 8 and $a_8 = 24$

Now, to find A_1, which is the second term of the sequence, we need to find the common difference first.

$$a_8 = 24$$

$\Rightarrow \quad a + 7d = 24$

$\Rightarrow \quad 3 + 7d = 24$

$\Rightarrow \quad d = 3$

$\therefore \quad A_1 = a + d = 3 + 3 = 6$

$A_2 = a + 2d = 3 + 6 = 9$

$A_3 = a + 3d = 3 + 9 = 12$

$A_4 = a + 4d = 3 + 12 = 15$

$A_5 = a + 5d = 3 + 15 = 18$

and $\quad A_6 = a + 6d = 3 + 6 \times 3 = 21$

Hence the six numbers between 3 and 23 are 6, 9, 12, 15, 18 and 21.

We may also observe that in general, an arithmetic mean An = a + nd; where a is the first term and d is the common difference.

Illustration 15: If $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the A.M. between 'a' and 'b', then find the value of 'n'.

Solution: Since $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ in the A.M. between 'a' and 'b',

then, $\frac{a^n + b^n}{a^{n-1} + b^{n-1}} = \frac{a+b}{2}$

$\Rightarrow 2(a^n + b^n) = (a^{n-1} + b^{n-1})(a + b)$

$\Rightarrow 2a^n + 2b^n = a^n + ab^{n-1} + a^{n-1} b + b^n$

$a^n + b^n = ab^{n-1} + a^{n-1} b$

$\Rightarrow \quad a^n - a^{n-1} b = ab^{n-1} - b^n$

$\Rightarrow \quad a^{n-1} (a - b) = b^{n-1} (a - b)$

$\Rightarrow \quad a^{n-1} = b^{n-1} \quad (\text{as } a - b \neq 0)$

$$\Rightarrow \quad \left(\frac{a}{b}\right)^{n-1} = 1$$

$$\Rightarrow \quad n - 1 = 0$$

$$\Rightarrow \quad n = 1$$

Check Your Progress 1

Multiple Choice Questions:

Choose the correct answer from the given four options.

1. If for a sequence $S_n = 2n - 3$, then the common difference is;

   (a) –1 (b) 2

   (c) –2 (d) 3

2. The number of integers from 100 to 500 that are divisible by 5 are:

   (a) 80 (b) 81

   (c) 75 (d) none of these

3. The number of two digit numbers divisible by 6 are:

   (a) 24 (b) 14

   (c) 15 (d) 20

4. If the fourth term of an A.P. is 4, then the sum of its 7 terms is:

   (a) 28 (b) 26

   (c) 32 (d) none of these

5. Which of the following terms are not the term of the A.P. –3, –7, –11, –15, ... –403, ..., –799.

   (a) –500 (b) –399

   (c) –503 (d) –51

Short Answer Type Questions:

6. If each term in a given A.P. is doubled, then is the new sequence obtained an A.P. If yes, then find its common difference.

7. Find the middle term in the A.P. 20, 16, 12, ..., –176.

8. In an A.P., if $a_4 : a_7 = 2 : 3$, then find $a_6 : a_8$

9. Find the sum of integers from 100 to 500 that are divisible by 2 and 3.

10. Find the sum of 20 terms of the A.P. whose n^{th} terms is $2n + 1$.

Long Answer Type Questions:

11. Inert 'n' arithmetic means between 1 and 31 such that the ratio of the 7^{th} mean and the $(n-1)^{th}$ mean is 5 : 9. Find the value of n and the resulting A.P.

12. Find the sum of the following series.

    (a) 4 + 7 + 10 + ... to 100 terms

    (b) $1 + \frac{4}{3} + \frac{5}{3} + 2 + \ldots$ to 19 terms

    (c) 0.5 + 0.51 + 0.52 ..... to 1000 terms

13. If the sum of term is an A.P. is 72, find the number of terms given that the first term of the sequence is 17 and common difference is –2.

14. If the sum of first p terms of an A.P. is equal to the sum of the first q terms, then proven that the sum of the first (p + q) terms is zero.

15. Find the first negative term in the given A.P..

    $19, \frac{91}{5}, \frac{87}{5}, \ldots$

16. If p^{th}, q^{th} and r^{th} terms of an A.P. are a, b, c respectively, then prove that.

    $(a - b)r + (b - c)p + (c - a)q$

17. Find the middle terms in the A.P. 5, 8, 11, 14,..... whose last term is 95.

18. In an A.P., if p^{th} terms is $\frac{1}{q}$ and q^{th} term is $\frac{1}{p}$ then prove that the sum of first pq terms is $\frac{1}{2}(pq+1)$, where $p \neq q$

19. The sum of 'n' term of two A.P.s are in the ratio 5n + 4 : 9n + 6. Find the ratio of the their 18^{th} terms.

20. On a certain day in a hospital, during covid crisis, the patients in the OPD were 1000. Due to efforts of the doctors and health care warriors and precautions taken by general public numbers declined by 50 per day. As per the decline in the number of patients, do you think that there would be a day with no patients in the OPD? If yes, which day would it be from the day when there were 1000 patients.

5.4 Geometric Progression (GP)

5.4.1 Definition

A sequence $a_1, a_2, a_3, \ldots a_n, \ldots$ is called a geometric progression, if each term is non-zero and $\frac{a_{k+1}}{a_k}$ = r (constant) for $k \geq 1$.

By letting $a_1 = a$, we obtain the geometric progression a, ar, ar^2, ar^3, ..., where a is called the first term and r is called the common ratio of the G.P.

Thus we define, GP is sequence in which each term except the first is obtained by multiplying the previous term by a non-zero constant. The first term is denoted by a and the constant number by which we multiply any term to obtain the next term is called common ratio and is denoted by r.

Some example of GP are:

(i) 3, 6, 12, 24, ... Here, a = 3, r = $\frac{a_2}{a_1} = \frac{6}{3}$ = 2

(ii) $\frac{1}{2}, -\frac{1}{6}, \frac{1}{18}, \ldots\ldots$ Here, a = $\frac{1}{2}$, r = $\frac{a_2}{a_1} = -\frac{1}{3}$

5.4.2 General Term of a G.P.

Let a be the first term and the r be the common ratio of a G.P.

Hence the G.P. is a, ar, ar^2, ar^3, ar^4, ...

also

$a_1 = a = ar^{1-1}$

$a_2 = ar = ar^{2-1}$

$a_3 = ar^2 = ar^{3-1}$

$a_4 = ar^3 = ar^{4-1}$

Do you see a pattern. What will be the 17th term.

$a_{17} = ar^{17-1}$

Therefore continuing the pattern we get

$a_n = ar^{n-1}$

Thus, a, GP can be written as a, ar, ar^2, ar^3, ...ar^{n-1} if it is finite or a, ar, ar^2, ...ar^{n-1} ..... if it is infinite. The corresponding geometric series are $a + ar + ar^2 + ... + ar^{n-1}$ and $a + ar + ar^2 + ... + ar^{n-1} + ...$ respectively.

Illustration 16 : Find the 10th and nth terms of the G.P.

2, 4, 8, 16, ...

Solution: Here a = 2, $r = \frac{a_2}{a_1} = \frac{4}{2} = 2$

Thus, $a_{10} = ar^9 = 2(2)^9 = 2^{10} = 1024$

and $a_n = ar^{n-1} = 2\,(2^{n-1}) = 2^n$

5.4.3 Selection of Terms of a G.P.

Sometimes, we require only certain number of terms of an A.P. The following ways of selecting terms are often convenient to find the solution of the problem:

Table 1.3.3

Number of Terms	Terms in a G.P.	Common Ratio
3	$\frac{a}{r}, a, ar$	r
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$	r^2
5	$\frac{a}{r^2}, \frac{a}{r}, \mathrm{a}, ar, ar^2$	r
6	$\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$	r^2

Please note that in case of odd number of terms, the middle terms is 'a' and common ratio is 'r' while in case of an even number of term two middle terms are $\frac{a}{r}, ar$ and the common ratio is r^2

Illustration 17: The third term of a G.P. is 12. Find the product of its first five terms.

Solution: Let the first term and common ratio of the GP be a and r respectively.

Given that, $t_3 = ar^2 = 12$ (1)

Now, $t_1\, t_2\, t_3\, t_4\, t_5 = a\,(ar)\,(ar^2)\,(ar^3)\,(ar^4)$

$= a^5 r^{\frac{4\times5}{2}}$

$= a^5r^{10} = (ar^2)^5$

$= (12)^5$ Using (1)

$= 248832$

Illustration 18: If the m^{th} and n^{th} terms of a G.P are n and m respectively, show that its $(m+n)^{th}$ term is $\left(\frac{n^m}{m^n}\right)^{\frac{1}{m-n}}$

Solution: Let a be the first term and r the common ratio of the G.P

$t_m = ar^{m-1} = n \qquad (1)$

$t_n = ar^{n-1} = m \qquad (2)$

Dividing (1) by (2) we get

$$\frac{ar^{m-1}}{ar^{n-1}} = \frac{n}{m} \Rightarrow (r)^{m-n} = \frac{n}{m}$$

$$\Rightarrow r = \left(\frac{n}{m}\right)^{\frac{1}{m-n}} \qquad (3)$$

Now, $t_{m+n} = ar^{m+n-1}$

$= ar^{m-1} \times r^n$

$= n \times \left[\left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right]^n$ Using (iii)

$$= \frac{n^{1+\frac{n}{m-n}}}{m^{\frac{n}{m-n}}} = \frac{n^{\frac{m}{m-n}}}{m^{\frac{n}{m-n}}} = \left(\frac{n^m}{m^n}\right)^{\frac{1}{m-n}}$$

Hence proved

5.4.4 Sum of n Terms of a G.P

Let the first term of a G.P be a and the common ratio be r. Let S_n denote the sum of first n terms of the G.P. Then

$S_n = a + ar + ar^2 + ... + ar^{n-1} \qquad (1)$

If $r \neq 1$, multiplying equation (1) by r, we get

$rS_n = ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n$ (2)

Subtracting (2) from (1), we get

$(1-r)\ S_n = a - ar^n$

$\Rightarrow S_n \frac{a(1-r^n)}{1-r}$ or $\frac{a(r^n-1)}{r-1}$

Note: If r = 1, we obtain

$S_n = a + a + a + ...+ a$ (n terms)

i.e., $S_n = na$

Illustration 19: Find the sum of indicated number of terms in the following G.P.

(i) 3, 6, 9, ...10 terms, n terms

(ii) 1, –3, 9, –27, ...8 terms, p terms

Solution:

(i) Here a = 3, r = $\frac{6}{3} = 2$

$S_n = \frac{a(r^n-1)}{r-1} = 3\ (2^n - 1)$

$S_{10} = \frac{3[2^{10}-1]}{2-1} = 3 \times 1023 = 3069$

(ii) Here a = 1, r = –3

$S_n = \frac{a(r^n-1)}{r-1} = \frac{1((-3)^n-1)}{-3-1}$

$S_n = \frac{(-3)^n-1}{-4}$

So, $S_p = \frac{1-(-3)^p}{4}$

and $S_8 = \dfrac{1-(-3)^8}{4}$ = – 1640

Illustration 20: How many terms of the G.P. 32, 16, 8, ... are needed to give the sum $63\dfrac{3}{4}$?

Solution: Let n terms of the G.P. be needed to give the sum $63\dfrac{3}{4}$ or $\dfrac{255}{4}$

Now, a = 32, r = $\dfrac{16}{32} = \dfrac{1}{2}$ and $S_n = \dfrac{255}{4}$

$\because S_n = \dfrac{a(1-r^n)}{1-r}$

$\therefore \dfrac{255}{4} = \dfrac{32\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}}$

$\Rightarrow \dfrac{255}{4} = 64\left(1-\left(\dfrac{1}{2}\right)^n\right)$

$\Rightarrow \left(\dfrac{1}{2}\right)^n = 1 - \dfrac{255}{256}$

$\Rightarrow \left(\dfrac{1}{2}\right)^n = \left(\dfrac{1}{2}\right)^8 \Rightarrow n = 8$

Illustration 21: The sum of three consecutive terms of a G.P. is 26 and their product is 216. Find the common ratio and the terms.

Solution: Let three consecutive terms of the G.P. be

$\dfrac{a}{r}$, a and ar

Then, $\left(\dfrac{a}{r}\right)(a)(ar) = 216 \Rightarrow a^3 = 216 = 6^3$

i.e., $a = 6$

Since, $\frac{a}{r} + a + ar = 26$

$\Rightarrow 6\left[\frac{1}{r}+1+r\right] = 26$

$\Rightarrow 3r^2 - 10r + 3 = 0$

$\Rightarrow (3r-1)(r-3) = 0$

$\Rightarrow r = \frac{1}{3}, 3$

If $r = \frac{1}{3}$, the numbers are 18, 6, 2

If r = 3, the numbers are 2, 6, 18.

Illustration 22: Find the sum of the sequence

4, 44, 444, ... to n terms

Solution: Let S_n be the sum of n terms of the sequence.

Observe carefully it is not a G.P., but can be converted into one as shown below:

$S_n = 4 + 44 + 444 + ...$n terms

$S_n = 4\ (1 + 11 + 111 + ...$n terms)

$S_n = \frac{4}{9}\ (9 + 99 + 999 + ...$to n terms)

$S_n = \frac{4}{9}\ ((10-1) + (10^2-1) + (10^3-1) + ...$n terms))

$S_n = \frac{4}{9}\ ((10 + 10^2 + 10^3 + ...$n terms) $- (1+1+1+.....$n terms))

$S_n = \frac{4}{9}\left[\frac{10\left(10^n-1\right)}{10-1}-n\right]$

$$S_n = \frac{4}{9}\left[\frac{10}{9}\left(10^n - 1\right) - n\right]$$

Illustration 23: An insect population is growing in such a way that each generation is 2.5 times as large as the previous one. If there are 10,000 insects in the first generation, how many are there in the 5^{th} generation.

Solution: Here, a = 10000 and r = 2.5

Number of insects in the 5^{th} generation is given by

$t^5 = ar^4$

$= 10000 \times (2.5)^4$

$= 390625$

Illustration 24: The population of a country is 114 million at present. If the population is growing at the rate of 10% per year. What will be the population of the country in 5 years?

Solution: The population of different consecutive years forms a G.P. A growth of 10% per year, when added to the previous year makes value of r = 110% = 1.1

The value of a_1 = 114 million and n = 5.

Therefore the population of the country in 5 years is given by:

$a_5 = a_1 r^4$

$= 114 \times (1.1)^4$

$= 114 \times 1.4641$

$= 166.9074 = 166.91$ million (app.)

Illustration 25: A manufacturer reckons that the value of a machine, which costs him Rs. 56200, will depreciate each year by 20%. Find the estimated value at the end of 3 years.

Solution: The value of the machine in various years will form a G.P. with first term a = Rs. 56200 and common ratio, r = 80% = 0.8 (a depreciation of 20% when subtracted from the previous year).

The estimated value at the end of 3 years is given by its value in the 4th year.

So, $a_4 = ar^3 = 56200 \times (0.8)^3$

$= 56200 \times 0.512$

$= 28,774.4$

Hence the value of the machine at the end of 3 years is Rs.28774.4

Illustration 26: A man saves ₹ 500 in the first month and in successive months he saves twice as much as in the previous month. This process continued for 6 months. From the seventh month and onwards he is able to save Rs. 500 less than previous month. Find his total savings for the year.

Solution: For first 6 months the savings forms a G.P. with first term 500 and common ratio r = 2.

∴ Total savings in the first six months.

$$S_6 = \frac{a(r^n-1)}{r-1} = \frac{500(2^6-1)}{2-1}$$

= 500 × 63 = Rs. 31,500

During the next 6 months the savings forms an A.P. with first term ₹ 500 less than the savings of the 6th month i.e., $a_6 = ar^5 = 500\ (2)^5$

So the first term of the AP is a = 16000

Hence for next 6 months his total savings will be

$$S_{\text{next }6} = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$= \frac{6}{2}\left[2\times 16000 + 5\text{x} - (500)\right]$$

$= \frac{6}{2}\left[32000 - 25000\right] = 21000$

So the total savings for the year

= Rs. 31,500 + Rs. 21,000

= Rs. 52,500

5.4.5 Geometric Mean

The geometric mean of two positive real numbers *a* and *b* is the number $G = \sqrt{ab}$

Therefore the geometric mean of 4 and 9 is $\sqrt{4 \times 9} = 6$. We observe that 4,6,9 are consecutive terms of a GP. This leads to a generalisation of the concept of geometric mean of two numbers.

Given any two positive numbers a and b, we can insert as many numbers as we want between them to make the resulting sequence a G.P.

Illustration 27: Insert 3 numbers between 1and 256 so that the resulting sequence is a G.P.

Solution : Let G_1, G_2, G_3, be three numbers between 1 and 256 such that 1, G_1, G_2, G_3, 256 forms a G.P. Now 256 is the (3+2) i.e., 5th term of the GP having first term 1

Let r be the common ration

So t_5 = 256 = ar^4=256=$1r^4$=256

$\Rightarrow$ r=4 or -4

If r=4, the G.P. is 1,4,16,64,256 and

If r =-4, the G.P. is 1,-4,16,-64,256

So the three numbers are 4,16,,64 or -4,16,-64.

5.5 Relationship between AM and GM

Let a and b be two given positive real numbers.

Also let their AM and GM be A and G respectively.

Then, $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

Thus, we have

$$A - G = \frac{a+b}{2} - \sqrt{ab}$$

$$= \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} \geq 0 \qquad (1)$$

From (1), we obtain $A - G \geq 0$, i.e., $A \geq G$

Note:

1. A = G holds if a = b
2. For three positive real numbers a, b and c

$$\frac{a+b+c}{3} \geq (abc)^{\frac{1}{3}} \qquad (\because \text{AM} \geq \text{GM})$$

Likewise the result can be generalised to n non-negative real numbers.

Illustration 28: If AM and GM of two positive numbers x and y are 13 and 12 respectively, find the numbers.

Solution: Given

$$\text{AM} = \frac{x+y}{2} = 13 \quad \Rightarrow \quad x + y = 26 \qquad (1)$$

$$\text{GM} = \sqrt{xy} = 12 \quad \Rightarrow \quad xy = 144 \qquad (2)$$

Substituting value of x+y and xy from (1) and (2) in the identity,

$$(x + y)^2 - (x - y)^2 = 4xy$$

We get, $26^2 - (x - y)^2 = 4 \times 144$

$\Rightarrow \quad (x - y)^2 = 100$

$\Rightarrow \quad x - y = \pm 10 \qquad (3)$

Solving (1) and (3), we obtain

$x = 18, y = 8$ or $x = 8, y = 18$

Thus the numbers x and y are 18, 8 or 8, 18 respectively.

Illustration 29: For any two positive real numbers a and b, prove that $\frac{a}{b}+\frac{b}{a} \geq 2$

Solution: Let $\frac{a}{b}$ = x and $\frac{b}{a}$ = y

Now, $\frac{x+y}{2} \geq \sqrt{xy}$ $\quad (\because$ AM $\geq$ GM)

$$\therefore \quad \frac{\frac{a}{b}+\frac{b}{a}}{2} \geq \sqrt{\frac{a}{b}\times\frac{b}{a}}$$

$$\Rightarrow \quad \frac{a}{b}+\frac{b}{a} \geq 2$$

Illustration 30: If $x \in R$, find the minimum value of $3^x + 3^{(1-x)}$.

Solution: Since AM $\geq$ GM

Therefore, $\frac{3^x + 3^{1-x}}{2} \geq \sqrt{3^x \times 3^{1-x}}$

$$\Rightarrow \quad \frac{3^x + 3^{1-x}}{2} \geq \sqrt{3}$$

$\Rightarrow \qquad 3^x + 3^{(1-x)} \geq 2\sqrt{3}$

Hence the minimum value of $3^x + 3^{1-x}$ is $2\sqrt{3}$

Illustration 31: If a, b, c and d are four distinct positive numbers in G.P then prove that $a + d \geq b + c$.

Solution:

$\because$ a, b, c, d are in GP

$\therefore$ a, b, c are in GP

$\therefore$ $a\,c = b^2$

Now for a and c,

$$\frac{a+c}{2} \geq \sqrt{ac} \qquad (\because \text{ AM} > \text{GM})$$

$$\Rightarrow \quad \frac{a+c}{2} \geq b \qquad (\because \; ac = b^2)$$

$\Rightarrow a + c > 2b \qquad (1)$

Again a, b, c, d are in GP

$\therefore$ b, c, d are in GP

$\therefore$ $c\,d = c^2$

Now for numbers b and d,

$$\therefore \frac{b+d}{2} \geq \sqrt{bd} \qquad (\because \text{ AM} > \text{GM})$$

$$\Rightarrow \frac{b+d}{2} \geq c \qquad (\because \; bd = c^2)$$

$\Rightarrow b + d \geq 2c \qquad (2)$

Adding (1) and (2) we get

$$a + c + b + d \geq 2\,(b + c)$$

$\Rightarrow \quad a + b + c + d \geq 2b + 2c$

$\Rightarrow \quad a + d \geq b + c$

Illustration 32: For positive real numbers a, b and c if a + b + c = 18, find the maximum value of abc?

Solution: Since, AM $\geq$ GM

Therefore, $\frac{a+b+c}{3} \geq (abc)^{\frac{1}{3}}$

$\Rightarrow \frac{18}{3} \geq (abc)^{\frac{1}{3}}$

$\Rightarrow (abc)^{\frac{1}{3}} \leq 6$

$\Rightarrow abc \leq 6^3$

$\Rightarrow abc \leq 216$

Hence the maximum value of abc is 216.

5.6 Infinite Geometric Progression

Infinite Geometric Progression is given as a, ar, ar^2, ar^3, ...

We know sum of GP a, ar, ar^2, ar^3, ... to n terms is given by

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{a}{1-r} - \frac{a}{1-r}(r^n)$$

If $|r| < 1$, as $n \to \infty$, $r^n \to 0$

Therefore, $S_n \to \frac{a}{1-r}$ as $n \to \infty$

Symbolically if sum to infinity is denoted by S_∞ or S

Then, S = S_∞ = $\frac{a}{1-r}$

For example:

(i) $1+\frac{1}{3}+\frac{1}{3^2}+.....=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}$

(ii) $1-\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+.....=\frac{1}{1-\left(-\frac{1}{2}\right)}=\frac{2}{3}$

Illustration 33: Find the sum to infinity of the G.P.

(i) 10, –8, 6.4, ...

(ii) a, br, ar^2, br^3, ar^4, br^5, ...; $|r|<1$

Solution:

(i) $\because |r|<1$,

$\therefore S_\infty = \frac{a}{1-r}=\frac{10}{1-\left(-\frac{4}{5}\right)}=\frac{10}{\frac{9}{5}}=\frac{50}{9}$

(ii) $S_\infty = (a + ar^2 + ar^4 + ...) + (br + br^3 + br^5 + ...)$

$= a(1 + r^2 + r^4 + ...) + br(1+ r^2 + r^4 + ...)$

$= a\left(\frac{1}{1-r^2}\right)+br\left(\frac{1}{1-r^2}\right)$

$= \frac{a+br}{1-r^2}$

Illustration 34: Represent the following as rational number

$0.3\overline{4}$

Solution: Let S = $0.3\overline{4}$

S = 0.3 + 0.04 + 0.004 + ...

$$S = \frac{3}{10} + \left(\frac{4}{100} + \frac{4}{1000} + ...\right)$$

$$S = \frac{3}{10} + \frac{4}{100}\left(1 + \frac{1}{10} + \frac{1}{10^2} + ...\right)$$

$$S = \frac{3}{10} + \frac{4}{100}\left(\frac{1}{1 - \frac{1}{10}}\right)$$

$$S = \frac{3}{10} + \frac{4}{100} \times \frac{10}{9} = \frac{3}{10} + \frac{2}{45} = \frac{31}{90}$$

Illustration 35: A substance initially weighing 64 grams is decaying at a rate such that after 8 hours there are only 32 grams remaining. In another 4 hours there are only 16 grams left, and after 2 more hours only 8 grams remain. How much time passes before none of the substance remains?

Solution: The weights of substance forms a GP given by 64 g, 32 g, 16 g, 8 g, ... with common ratio $\frac{1}{2}$. It is clear that $a_n \to 0$ as $n \to \infty$.

So the whole of substance will decay in S = 8 + 4 + 2 + ... hours which is again a G.P. with first term 8 and r = $\frac{1}{2}$.

Hence $S = \frac{8}{1 - \frac{1}{2}} = \frac{8}{\frac{1}{2}}$ = 16 hours.

Illustration 36: A pendulum swings through an arc of 25 cm. On each successive swing, the pendulum covers an arc equal to 90% of the previous swing. Find the length of the arc on the sixth swing and the total distance the pendulum travels before coming to rest.

Solution: Here a = 25 cm and r = $\frac{9}{10}$

The length of the arc on 6th swing = ar^5

= 25 × (0.9)5

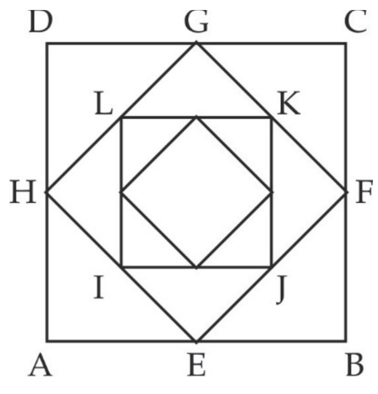
= 14.76 cm

Let the pendulum comes to rest after travelling a total distances

S = $\frac{a}{1-r} = \frac{25}{1-0.9} = \frac{25}{\frac{1}{10}}$ = 250 cm

Illustration 37: A square is drawn by joining the mid-points of the sides of a square. A third square is drawn inside the second square by joining the mid-points of the second square and the process is continued indefinitely. If the side of the original square is 8 cm, find the sum of the areas of all the square thus formed.

Solution: Given that AB = 8 cm,



HE = $\sqrt{AE^2 + AH^2} = \sqrt{4^2 + 4^2} = 4\sqrt{2}$

Also LI = $\sqrt{HI^2 + HL^2} = \sqrt{\left(2\sqrt{2}\right)^2 + \left(2\sqrt{2}\right)^2}$

= 4

and so on

Sum of the areas of all the squares is given below:

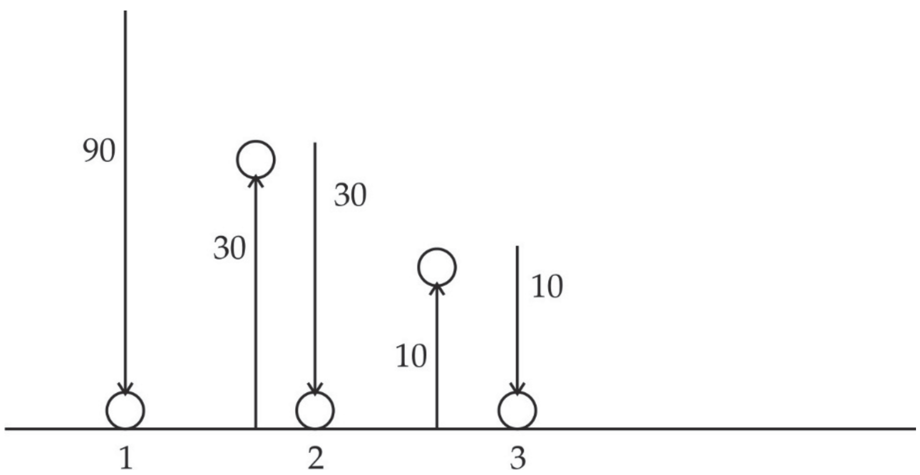
$$S_\infty = 8^2 + \left(4\sqrt{2}\right)^2 + (4)^2 + \ldots..$$

$$= 64 + 32 + 16 + \ldots..$$

$$= \frac{64}{1-\frac{1}{2}} = 128 \text{ cm}^2$$

Illustration 38: A ball is dropped from a height of 90 feet and always rebounds one-third the distance from which it falls. Find the total vertical distance the ball travelled when it hits the ground for the 3rd time. Also find the total distance travelled by the ball before it comes to rest.

Solution: First the ball falls 90 feet. Then the ball rebounds $\frac{1}{3}$ rd the distance i.e., 30 feet and falls 30 feet to hit the ground for the second time. On next rebound the ball rises 10 feet and falls back 10 feet to hit the ground for 3rd time as shown in the following figure.



Total distance travelled by the ball when it hits the ground for the 3rd time

$$= 90 + 2\ (30) + 2\ (10) = 170 \text{ feet}$$

Total distance travelled before coming to rest:

$$90 + 2\left(30 + 10 + \frac{10}{3} + ...\right) = 90 + 2\left(\frac{30}{1-\frac{1}{3}}\right) = 180 \text{ feet}$$

Check your Progress - 2

1. Find the indicated terms in each of the Geometric progressions given below:

   (i) 4, 12, 36, ...; 5^{th} term

   (ii) $3, -1, \frac{1}{3}, -\frac{1}{9}$, ...; 4^{th} term, nth term

2. Which term of the following sequences.

   (i) 5, 10, 20, 40, ... is 5120

   (ii) $2, 2\sqrt{2}, 4$, ... is 128

   (iii) $2, 1, \frac{1}{2}, \frac{1}{4}$, ... is $\frac{1}{128}$

3. Find the sum to indicated number of terms in each of the geometric progressions

   (i) $\sqrt{3}, 3, 3\sqrt{3}$, ... 6 terms

   (ii) 0.15 + 0.015 + 0.0015 + ... 20 terms.

4. Evaluate $\sum_{R=1}^{10}(3+2^k)$

5. The sum of the first two terms of a G.P. is 36 and the product of first term and third term is 9 times the second term. Find the sum of first 8 terms.

6. Find the sum to n terms of the sequence.

   7, 77, 777, 7777, ...

7. The sum of first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.

8. Find four numbers forming a G.P. in which the third term is greater than the first term by 9, and the second term is greater than the 4th by 18.

9. Insert 6 geometric means between 27 and $\frac{1}{81}$.

10. If the AM of two unequal positive real numbers a and b (a > b), be twice as much as their G.M., show that

$$a : b = (2 + \sqrt{3}) : (2 - \sqrt{3})$$

11. If a, b, c, d are in G.P., show that

(i) $a^2 + b^2, b^2 + c^2, c^2 + d^2$ are n G.P.

(ii) $\frac{1}{a^2+b^2}, \frac{1}{b^2+c^2}, \frac{1}{c^2+d^2}$ are in G.P.

12. Let S be the sum, P the product and R the sum of reciprocals of n terms of a G.P. Prove that $P^2 R^n = S^n$.

13. What will Rs. 5000 amount to in 10 years after it is deposited in a bank which pays annual interest of 8% compounded annually?

14. If the first and the nth term of a G.P. are a and b, respectively, and if P is the product of n terms, prove that $P^2 = (ab)^n$.

15. A certain type of bacteria doubles its population every 20 minutes. Assuming no bacteria die, how many bacteria will be there after 3 hours if there are 1 million bacteria at present.

16. One side of an equilateral triangle is 24 cm. The mid points of its sides are joined to form another triangle whose mid points are joined to form yet another triangle and so on. This process continues indefinitely. Find the sum of the perimeters of all the triangles.

17. After striking a floor a certain ball rebounds $\frac{4}{5}^{th}$ of the height from which it has fallen. If the ball is dropped from a height of 240 cm, find the total distance the ball travels before coming to rest.

18. An object decelerates such that it travels 60 m during the first second, 20 m during the second and $6\frac{2}{3}$ m during the third second. Determine the total distance the object travels before coming to rest.

19. Suppose a person mails a letter to five of his friends. He asks each one of them to mail it further to five additional friends with instruction that they move the chain further. Assuming the chain is not broken and no person receives the mail more than once, determine the amount spent on postage when the 8^{th} set of letters is mailed, if cost of postage of each letter is 50 paisa.

20. Due to reduced taxes an individual has an extra Rs. 30,000 in spendable income. If we assume that an individual spends 70% of this on consumer goods and the producers of these goods in turn spends 70% on consumer goods and this process continues indefinitely. What is the total amount spent on consumer goods.

21. A machine depreciates in value by one-fifth each year. If the machine is now worth Rs. 51000, how much will it be worth 3 years from now.

22. The sum of an infinite G.P. is 3 and the sum of the squares of its terms is also 3, than its first term and common ratio are:

 (i) $1, \frac{1}{2}$

 (ii) $\frac{1}{2}, \frac{3}{2}$

 (iii) $\frac{3}{2}, \frac{1}{2}$

 (iv) $1, \frac{1}{4}$

23. An antiques present worth is Rs. 9000. If its value appreciates at the rate of 10% per year. Its worth 3 years from now is:

 (i) Rs. 6561

 (ii) Rs. 10,890

(iii) Rs. 11,979

(iv) Rs. 12,000

24. Venessa invests Rs. 5000 in a bond that pays 6% interest compounded semi-annually. The value of the bond in rupees after 5 years is:

(i) $5000\ (1.06)^5$

(ii) $5000\ (1.03)^5$

(iii) $5000\ (1.06)^{10}$

(iv) $5000\ (1.03)^{10}$

Illustration - 39

Economy Stimulation

The government through a subsidy program infuses 10 lakh crore to boost the sagging economy. If we assume each state or UT spends 80% of what is received on infrastructure, health and industry and in turn each institution or agency spends 80% of what is received and so on. How much total increase in spending results from this government action?

Solution: According to the multiplier principle in economics, the effect of stimulus of 10 lakh crore on the economy is multiplied many times.

To find the amount spent if the process continues as given is obtained by applying infinite G.P. whose first term a_1 is the first amount spend i.e.,

$$a_1 = \frac{80}{100} \times 10 \text{ lakh crore} = 8 \text{ lakh crore}$$

Also, $r = 0.8$

Hence $S_\infty = 8 + 8\ (0.8) + 8\ (0.8)^2 + ...$

$$S_\infty = \frac{8}{1-0.8} = \frac{8}{0.2} = 40 \text{ lakh crore}$$

So an economic stimulus of 10 lakh crore would result in spending of about 40 lakh crore.

Illustration - 40

Facing the Pandemic (The Virus Spread)

The population of a certain city is 51,20,000. Let us assume 5000 persons in the city are infected by corona-virus. The virus is so contagious that the number of infected persons doubles every 12 days. If the rate of doubling remains constant then in how many days 50% of the population gets infected.

Solution: Here, the first term is the initial number of persons infected by the virus.

i.e., $a_1 = 5000$

Since the number of patients doubles every 12 days

So $r = 2$

If the required number of infected persons be denoted by a_n then

$$a_1 r^{n-1} = a_n$$

$$\Rightarrow 5000\,(2)^{n-1} = \frac{1}{2} \times 5120000$$

$$\Rightarrow 2^n = 1024 = 2^{10}$$

$$\Rightarrow n = 10$$

Number of days required to infect half the entire population is

$$= 12\,(n-1) = 12 \times 9 = 108 \text{ days}$$

Summary

- A sequence means arrangement of numbers in a definite order according to some rule. A sequence having countable number of terms is called finite sequence and a sequence is called infinite if it is not a finite sequence
- Let $a_1, a_2, a_3, \ldots$ be a given sequence, Then $a_1+a_2+a_3+\ldots$ is called a series

- An arithmetic progression (AP) is a sequence in which each term except the first is obtained by adding a constant number to the previous term. The first term is denoted by a, common difference by d and the last term by ℓ.
- The general term of the AP is given by an=a+(n-1) d
- The sum of n terms of the AP is given by

$$Sn = \frac{n}{2}\left[2a + (\mathrm{n}-1)\,\mathrm{d}\right] = \frac{n}{2}(\mathrm{a}+\mathrm{R})$$

- Arithmetic Mean of two positive real numbers a and b is the number $A = \frac{a+b}{2}$
- A sequence is said to be geometric progression (GP) if the ratio of any term to its preceding term is same throughout. The first term and common ratio of the GP is denoted by a and r respectively.

∴ The general term of teh GP is given by $a_n = ar^{n-1}$

∴ The sum sn of the first n terms of the GP is given by $S_n = \frac{a(r^n - 1)}{r-1} or \frac{a(1-r^n)}{1-r}$

- Relation between arithmetic mean A and Geometric mean G is given by

  A $\geq$ G
- Sum of infinite GP a, ar, ar^2,... where $|r| < 1$ is denoted by S_∞ sm and is given by $S_\infty = \frac{a}{1-r}$
- Application problems based on GP.

Solution to check your progress

Check your Progress - 1

1 (d) 2(b) 3(c) 4(a) 5(a) 6 Yes; common difference is twice the common difference of original sequence

7. a_{25} and a_{26} are the middle terms.

a_{25} = -76, a_{26} = -080

8. $a_6 : a_8 = 4 : 5$

9. 20100

10. 440

11. 14, 1, 3, 5, 7, 9, 11, ….., 31

12 5495

13 n=6, n=12

15 25 term = $\frac{-1}{5}$

17 50, the 16th term

19 179 : 321

20 on 21st day there would be no Covid Patient report in the hospital

Check your Progress-2

Q1 (i) 324 (ii) $-\frac{1}{9}, \frac{(-1)^{n-1}}{3^{n-2}}$

Q2 (i) 11 (ii) 13 (iii) 9

Q3 (i) $39+13\sqrt{3}$ ii $\frac{1}{6}\left[1-(0.1)^{20}\right]$

Q4 $28+2^{11}$

Q5 $\frac{3280}{81}$

Q6 $\frac{70}{81}(10^n-1)-\frac{7n}{9}$

Q7 $r=\frac{5}{2} or \frac{2}{5}; term\ are\ \frac{2}{5},1,\frac{5}{2}\ or\ \frac{5}{2},1,\frac{2}{5}$

Q8 3, -6, 12, -24

Q9 9, 3, 1, $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}$

Q13 Rs 5000 $(1.08)^{10}$

Q15 512 million

Q16 144 cm

Q17 21.6m

Q18 90^{m}

Q19 Rs 24,4140.60

Q20 Rs 70,000

Q21 Rs 26,112

Q22 (iii)

Q23 (iii)

Q24 (iv)

Practice Questions:

Multiple Choice Questions:

1) Find the sum of the G.P.: 8/10, 8/100, 8/1000, 8/10000, … to n terms.

   a) $\frac{-8}{9}\left(\frac{1}{10^n}-1\right)$

   b) $\frac{8}{81}\left(\frac{1}{10^n}-1\right)$

   c) $\frac{9}{8}\left(\frac{1}{10^n}-1\right)$

   d) $\frac{8}{90}\left(\frac{1}{10^n}-1\right)$

2) If the first term of a GP be 5 and common ratio is (-5) then which term is 3125.

   a) 6^{th}

b) 8^{th}

c) 5^{th}

d) 4^{th}

3) Which number should be added to the numbers 3, 8, 13 to make the resulting numbers should be GP.

a) 4

b) 2

c) 5

d) -2

4) If the third term of a G.P. is 6 then the product of its first 5 term is

a) 5^6

b) 6^5

c) 5^2

d) 6^2

5) If a, b and c are in A.P. as well as in G.P., then which of the following is true.

a) $a=b \neq c$

b) $a \neq b \neq c$

c) $a=b=c$

d) $a \neq b = c$

Problems:

1) You friend has invested in a 'Grow Your Money Scheme' that promises to return Rs 11,000 after a year if you invest 1,000 at the rate of 10% compounded annually. You are bit skeptic about the claim of this scheme. Your Friend shows the calculation done by the agent of the scheme as follows. Would you be willing to invest in this scheme? Explain.

Here is how I have done

Given: P = Rs. 1000

$r = 10\%$

$n = 1$ year

$A = P\,(1 + r)^t$

$= 1000\,(1 + 10)^1$

$= 1000 * 11$

2) Write the first four terms of a geometric series for which $S_8 = 39{,}360$ and $r = 3$. (Adopted from augusta.k.12)

3) Using Geometric series write 0.175175175175…. in fraction.

4) In a mock test of Sequence & Series, Rohan & Shweta have solved a question in the following manner. Find the 10^{th} term of the Geometric series 9,3,1, ……….

Shweta:

$r = \frac{3}{9}$ or

$a_{10} = ar^9 = 9(\frac{1}{3})^9$

$a_{10} = \frac{1}{2187}$

Roshan:

$r = \frac{9}{3}$ or $r = 3$

$a_{10} = ar^9 = 9(3)^9$

$a_{10} = 177147$

a) Who is correct explain your reasoning?

b) Can you guess the correct answer without solving? If yes what argument would you use?

5) On the first day, a music video of Arjit Singh was posted online, got 120 views in Delhi. The number of viewership gets increased by 5% per day. How many total views did the video get over the course of the first 29 days? Express your answer in exponential form. (From Delta Math)

6) In the book of Harry Potter and Deathly Hallows, Harry traced his family history and got to know that he is related with Peverell brothers. Let's assume that he had traced his family back for 15 generations starting with his parents, how many ancestors he had in total. Try this exercise at home. To make it more interesting you could draw the family tree with the photos of your relative. See if you have more relatives than Harry Potter. (Adopted from augusta.k12.va.us)

7) You and your sibling decide to ask for a raise in your pocket money from your Dad. Your Dad gives both of you a choice. You two could have 1000 Rs at once or can get Rs 2 on day one, Rs.4 on day two so on receiving twice as many rupees each day as the previous day, for 12 days. While you opted for getting Rs 1000 at once your brother opted for the later. Which one of you made a better decision and why?

8) If $a^x = b^y = c^z$ such that a, b and c are in GP and x, y and z are unequal positive integer then show that $\frac{2}{y} = \frac{1}{x} + \frac{1}{z}$.

9) A person sends a fake news on WhatsApp to 4 of his friends on Monday. Each of those friends forward the fake news on to the four of their friends on Tuesday. Each person who receives the fake news on Tuesday send it four more people by Wednesday and this process goes on for a week. After a week a leading news agency verifies the news and finds it to be fake. Find how many people have received the fake news on WhatsApp

till then? Usually the fake news is shared and forwarded on a bigger scale. Try this question.

10) Three positive numbers form an increasing G.P. If the middle term of the series is doubled then the new numbers are in A.P. Find the common ratio of the G.P.

11) Priyanka invested Rs. 1300 in an account that pays 4% interest compounded annually. Assuming no deposits or withdrawals are made, find how much money she would have in the account 6 years after her initial investment. (Adapted from DeltaMath)

12) Dividend: A sum of money paid regularly by a company to its shareholders out of its profits.

A financial analyst is analyzing the prospects of a certain company. The company pays an annual dividend on its stock. A dividend of Rs. 500 has just been paid and the analyst estimates that the dividends will grow by 20% per year for the next 3 years, followed by annual growth of 10% per year for 2 years. (Adapted from Finance and Growth)

(a) Complete the following table:

Year	1	2	3	4	5
Dividend					

(b) Then calculate the total dividend that will be paid for the next five years.

13) In general, it has been assumed that compound interest is compounded once a year. In reality, interest may be compounded several times a

year, e.g. daily, weekly, quarterly, semi-annually or even continuously. The value of an investment at the end of m compounding periods is:

$$P_t = P_0(1 + r/m)^{m * t}$$

where m is the number of compounding periods per year and t is the number of years.

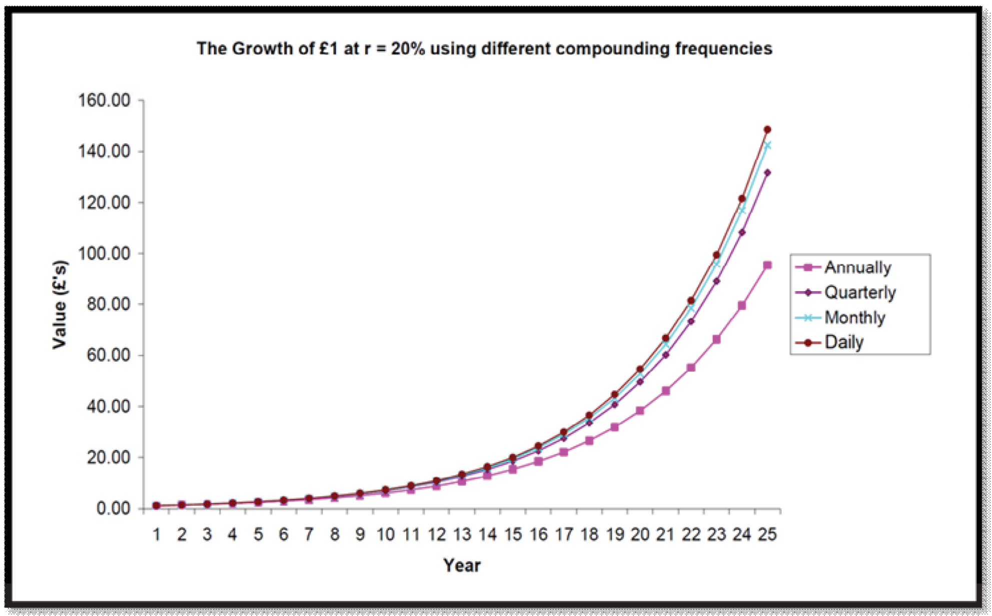
Using this information, solve the following problem:

(a) Rs.1,000 is invested for three years at 6% per annum compounded semi-annually. Calculate the total return after three years.

(b) What would the answer be if the interest was compounded annually?

(c) Using your answers of (a-b), what can you infer about the frequency of compounding and the size of the total return? (From Finance and Growth)

14) From this graph, what conclusion can be drawn regarding the frequency of compounding? (Adapted from Finance and Growth)



15) A small country emits 130,100 kilotons of carbon dioxide per year. In a recent global agreement, the country agreed to cut its carbon emissions by 3.1% per year for the next 3 years. In the first year by as per a special agreement, the country will keep its emissions at 130,000 kilotons and the

emissions will decrease 3.1% in each of the next two years. How many kilotons of carbon dioxide would the country emit over the course of the 3-year period? (Geometric Series- Deltamath.com)

16) https://www.khanacademy.org/math/geometry-home/geometry-volume-surface-area/koch-snowflake/v/koch-snowflake-fractal

17) A stock begins to pay dividends with the first dividend, one year from now, expected to be Rs. 100. Each year the dividend is 10% larger than the previous year's dividend. In what year the dividend paid is larger than Rs. 1000?

(Use concept of logarithm to solve the question)

Chapter VI

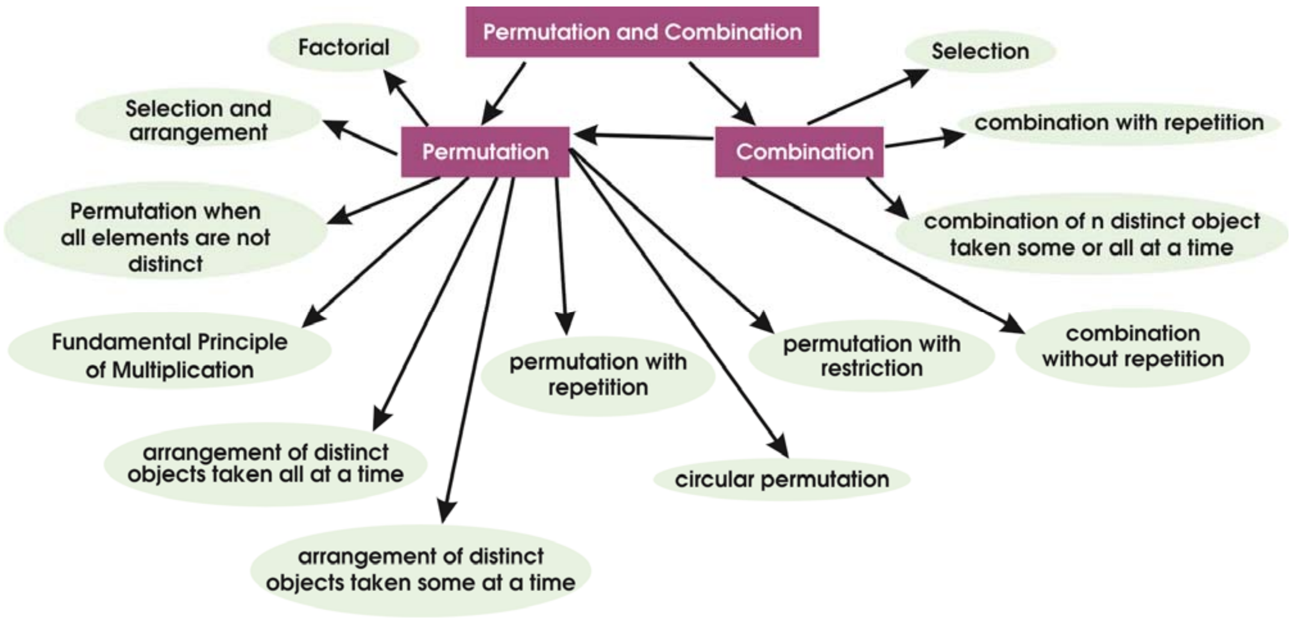
Permutations and Combinations

Learning Objectives:

After completion of this unit you will be able to:

- Explain the difference between permutation and combination
- Solve problems based on permutation and combination
- Apply the formulas and concepts learnt to find solution of application problems
- Appreciate how to count without counting.

Concept MapP



6.1 Introduction

Counting or enumeration is usually a common process when a student is studying arithmetic. But in later years very little attention is paid to counting when a student turns to difficult areas in mathematics, such as algebra, geometry and calculus.

A password on a computer system consists of eight characters. Each of these characters must be a digit or a letter of the alphabet. Each password must contain atleast one digit. How many such password are possible? The technique required to answer this and a wide variety of other counting problems will be introduced in this chapter.

Counting does not end with arithmetic. It has wide applications in probability, statistics, coding theory and analysis of algorithms.

Before we introduce the concept further we learn the meaning and notation of factorial.

6.2 Factorial

The product of first n natural numbers is called 'n factorial' and is denoted by n!

i.e., the product

$1 \times 2 \times 3 \times .... \times (n-1) \times n$ is denoted by n! and we read this symbol as 'n factorial'.

In particular

$$1 = 1!$$
$$1 \times 2 = 2!$$
$$1 \times 2 \times 3 = 3!$$
$$1 \times 2 \times 3 \times 4 = 4!$$
$$1 \times 2 \times 3 \times 4 \times 5 = 5! \qquad \text{and so on}$$

Thus $1 \times 2 \times 3 \times ... \times (n-1) \times n = n!$

Further we define $0! = 1$

Note: The factorial of a negative integer or a fraction is not defined.

We can write $7! = 7 \times 6! = 7 \times 6 \times 5! = 7 \times 6 \times 5 \times 4!$ and so on

Thus clearly, for a natural number n

$$n! = n\,(n-1)!$$
$$= n\,(n-1)\,(n-2)! \qquad \text{(provided } n > 2)$$
$$= n\,(n-1)\,(n-2)\,(n-3)! \qquad \text{(provided } n > 3)$$

and so on

Example 1: Evaluate (i) 5! (ii) 7! (iii) 7! – 5!

Solution:

(i) $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

(ii) $7! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040$

(iii) $7! - 5! = 7 \times 6 \times 5! - 5! = (42-1)\,5!$

$= 41 \times 5! = 4920$

Example 2: Compute (i) $\frac{11!}{9!}$ (ii) $\frac{13!}{(2!)(11!)}$

Solution:

(i) $\frac{11!}{9!} = \frac{11 \times 10 \times 9!}{9!} = 110$ (ii) $\frac{13!}{(2!)(11!)} = \frac{13 \times 12 \times (11!)}{2 \times (11!)} = 78$

Example 3: Find the LCM of 5!, 6! and 7!

Solution: LCM of 5!, 6! and 7!

$= \text{LCM of } 5!, 6 \times 5! \text{ and } 7 \times 6 \times 5!$

$= 7 \times 6 \times 5! = 7!$

Example 4: Express the following in factorial notation.

(i) $6 \times 7 \times 8 \times 9$

(ii) $4 \times 5 \times 6^2 \times 7 \times 8$

Solution:

(i) $6 \times 7 \times 8 \times 9 = \frac{1 \times 2 \times 3 \times 4 \times 5 \times (6 \times 7 \times 8 \times 9)}{1 \times 2 \times 3 \times 4 \times 5} = \frac{9!}{5!}$

(ii) $4 \times 5 \times 6^2 \times 7 \times 8 = 6 \text{ X } (4 \times 5 \times 6 \times 7 \times 8)$

$= 1 \times 2 \times 3 \times (4 \times 5 \times 6 \times 7 \times 8)$

$= 8!$

Example 5: If $\frac{1}{9!} + \frac{1}{10!} = \frac{x}{11!}$, find x

Solution: We have, $\frac{1}{9!} + \frac{1}{10+9!} = \frac{x}{11 \times 10 \times 9!}$

$$\Rightarrow \quad 1+\frac{1}{10}=\frac{x}{110}$$

$$\Rightarrow \quad \frac{11}{10}=\frac{x}{110} \text{ i.e., } x = 121$$

Example 6: Convert the following products into factorial notation

(i) 2.4.6.8... (2n)

(ii) 1.3.5... (2n–1)

Solution:

(i) 2.4.6.8 (2n)

= (2.1) × (2.2) × (2.3) × (2.4)x... × (2.n)

= 2^n (1.2.3.4...n)

= 2^n (n!)

(ii) 1.3.5... (2n –1)

$$= \frac{1.2.3.4.5...(2n-1)(2n)}{2.4.6...(2n)}$$

$$= \frac{(2n)!}{2^n[1.2.3...n]}=\frac{(2n)!}{2^n(n!)}$$

Example 7: Show that, 41! + 1 is not divisible by any number from 2 to 41.

Solution: Clearly 41! is divisible by every integer from 2 to 41. Thus 41! + 1 leaves 1 as remainder when divided by any integer from 2 to 41. Hence 41! + 1 is not divisible by any integer from 2 to 41.

Exercise 1.1

1. Evaluate

   (i) $6!$

   (ii) $\dfrac{20!}{18!}$

   (iii) $\dfrac{9!-8!}{7!}$

2. Is $4! + 5! = 9!$

3. Compute $\dfrac{n!}{(n-r)!}$ when

   (i) $n = 8, r = 2$

   (ii) $n = 12, r = 3$

4. If $\dfrac{1}{6!}+\dfrac{1}{7!}=\dfrac{x}{8!}$, find x

5. Evaluate $\dfrac{m!}{r!(n-r)!}$, when

   (i) $n = 13, r = 2$

   (ii) $n = 8, r = 5$

6. Show that $(n+2)\, n! = n! + (n+1)!$

7. Find n if

   (i) $(n+1)! = 20\,(n-1)!$

   (ii) $(n+2)! = 12\,(n!)$

8. Show that, $n\,(n-1)\,(n-2)\,...\,(n-r+1) = \dfrac{n!}{(n-r)!}$

9. If $\dfrac{n!}{2!(n-2)!}\div\dfrac{n!}{4!(n-4)!}=2,$ find the value of n.

Answers:

1. (i) 720 (ii) 380 (iii) 64

2. No

3.	(i) 56	(ii) 1320
4.	64	
5.	(i) 78	(ii) 56
7.	(i) 4	(ii) 2
9.	5	

6.3 Fundamental Principles of Counting

The study of permutation and combination begins with two basic principles of counting the rules of sum and product. We will begin with simple applications of these rules. While solving difficult problems, we decompose them into simple parts solve them and finally obtain the final answer.

Our first principle of counting can be stated as follows:

The Rule of Sum (**Fundamental Principle of Addition**) A job occurs if either event E occurs or event F occurs exclusively.

If event E can occur in m ways and event F can occur in n ways, then the job can occur in (m + n) ways.

This rule can be further extended

Example 8: A school library has 13 and 7 books available on financial mathematics written by Indian and foreign authors respectively. A student wants to learn more about the subject and has only 1 ticket to borrow the book. In how many ways the book can be borrowed.

Solution: By rule of sum (i.e., the fundamental principle of addition) a student can select the book in = 13+7 = 20 ways.

Example 9: A man is advised to go in for a knee replacement surgery. This facility is available in 5 private or 3 government hospitals. Alternatively he can go abroad where his son who is an orthopedic surgeon can operate. Find the number of options available to him.

Solution: By rule of sum (i.e., the fundamental principle of addition) there are = 5 + 3 + 1, i.e., 9 options available to the person.

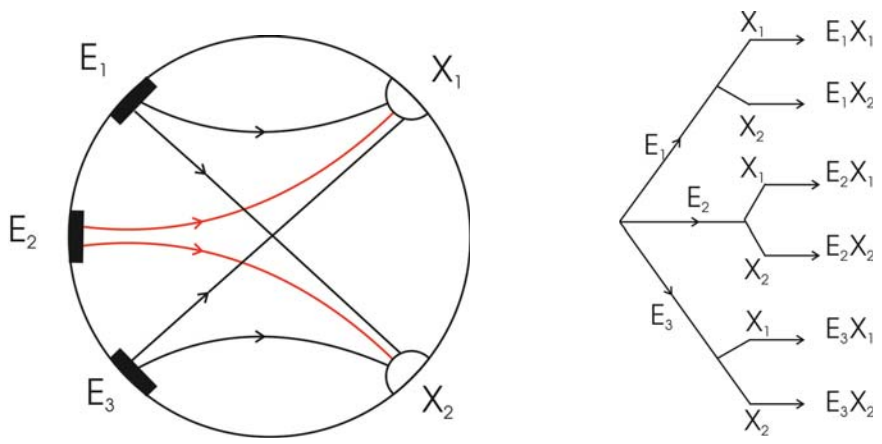
The solution of the following problem help us introduce second principle of counting.

Anirudh goes to a stadium to watch a cricket match. If there are 3 independent gates to enter the stadium and 2 independent gates to exit. List the different ways Anirudh can enter and exit the stadium if there is no restriction on the gate, one can use.

Let us name the three entrance gates as E_1, E_2, E_3 and the two exit gates as X_1, and X_2.

How many different pairs of entering and exciting the stadium are possible? There are 3 ways in which Anirudh can enter the stadium. For every choice of entering gate there are two choices of exiting. Therefore there are $3 \times 2 = 6$ different ways in which one can enter and exit the stadium.

Then the various possibilities of entering and exiting the stadium are illustrated in the following figure.

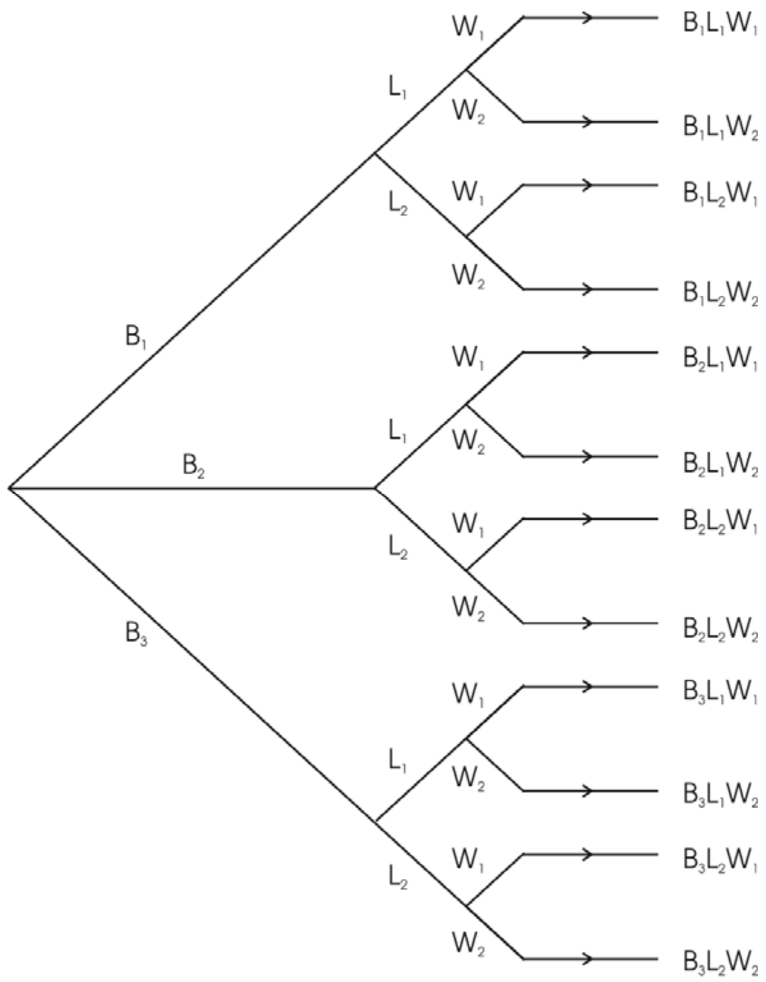


There are six possibilities as illustrated in the above figure.

Let us consider the following example to show that the rule can be extended to more than two tasks or events.

Reema goes to the market to buy school bag, lunch box and water bottle for her younger sister for the new academic session. The school bag can be chosen in 3 different ways. After the school bag is chosen, a lunch box can be chosen in 2 different ways. Hence, there are $3 \times 2 = 6$ pairs of school bag and lunch box to choose from. For each of these pairs a water bottle can be chosen in 2 different ways. Hence, there are $6 \times 2 = 12$ different ways in which, Reema can buy these items for her sister.

If we name the 3 school bags as B_1, B_2 and B_3, the two lunch boxes as L_1, L_2 and the two water bottles as W_1, W_2, then the various possibilities can be shown with the arrow diagram shown below.



Total = 12 possibilities

In fact the problem of the above type can be solved by applying the following principle known as fundamental principle of multiplication or the Rule of Product.

The Rule of Product (Fundamental Principle of Multiplication)

"If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of ways in which the two events can occur in the given order is m × n."

The above principle can be generalized for any finite number of events. For example, for 3 events, the principle is as follows:

"If an event can occur m in the different ways, following which another event can occur in n different ways, following which a third event can occur in p different ways, them the total number of ways of occurrence of the events in the given order is m x n x p.

In the first problem the number of ways of visiting the stadium consists of the occurrence of the following events in succession.

(i) the event of choosing the entrance gate i.e., 3

(ii) the event of choosing the exit gate i.e., 2

There are various possible ways in which one can enter and exit the stadium. We have to choose any one of the possible ways which can be done by counting the different ways of occurrence of the events in succession.

We observe the number of possible ways the product of number of ways of occurrence of the successive events.

So the number of possible ways = 3 × 2 i.e. = 6 ways

By similar argument, we can conclude that the number of possible ways of choosing a school bag, lunch box and water bottle is 3 × 2 × 2 = 12 ways.

Example 10: In a class there are 20 boys and 15 girls. The teacher wants to select either a boy or a girl to represent the class in a competition. In how many ways can this be done.

Solution: By fundamental principle of addition the teacher can select a boy or a girl in = 20 + 15 i.e. = 35 ways.

Example 11: The dramatics club of a college has selected six boys and five girls for a play. From this group the director can cast his leading couple (a boy and a girl) in how many ways?

Solution: The director can cast a boy for leading role in = 6 ways

The director can cast a girl for leading role in = 5 ways

By fundamental principle of multiplication the director can cast the couple in $6 \times 5 = 30$ ways.

Example 12: The chairs in an auditorium are to be labelled with a letter and a positive integer not exceeding 10. What is the largest number of chairs that can be put in the auditorium?

Solution: The letter can be assigned in 26 ways.

The number can be assigned in 10 ways.

The different ways the chair can be labelled is = 26×10 i.e., 260 ways

Hence the capacity of the auditorium if all possible labelled chairs are there is 260.

Example 13: How many different license plates are possible, if each plate contains a sequence of three letters starting with D followed by four digit - non zero number.

Solution: Since the number plate starts with D, so there is only one choice for the first letter. There are 26 choices for each of the remaining 2 letters and 10 choices for each of the four digits (i.e., 0 to 9)

By fundamental principle of multiplication the total possibilities are

$$= 1 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6760000$$

The sequence of letters followed by four zeros

$$= 1 \times 26 \times 26 \times 1 \times 1 \times 1 \times 1$$

$$= 676$$

The license plates with 4 digit non zero numbers

= 6760000 – 676

= 6759324

Example 14: An investment banker finalizes the list of specific entities worthy of investment in each of the three instruments mentioned below:

– 3 Public Ltd. companies for direct equity investment.

– 5 Mutual fund schemes

– 2 Banks for Fixed Deposit (F.D)

In how many ways the investment can be made if the board decides to.

(i) invest the entire fund in one entity (i.e., company, scheme or bank)

(ii) invest in one entity of each of the three instruments.

Solution:

(i) If the entire fund is invested in one entity.

By fundamental principle of addition (the rule of sum) the investment can be made in 3 + 5 + 2 = 10 ways

(ii) Investment in one entity of each instruments

By fundamental principle of multiplication or (the rule of product)

Public Ltd. company can be chosen in = 3 ways

Mutual Fund scheme can be selected in = 5 ways

Bank can be selected in = 2 ways

Total number of ways = $3 \times 5 \times 2 = 30$ ways

Example 15: How many three digit numbers can be formed using the digits 0, 1, 3, 5 and 8.

(i) If repetition of digits is not allowed

(ii) If repetition of digits is allowed.

Solution:

There be as many numbers as there are ways of filling 3 vacant places $\boxed{H}\boxed{T}\boxed{U}$ in succession by the digits. Since 0 cannot come in hundred's (run on) place so it can be filled in = 4 ways

The ten's place can be filled in (as 0 can come) = 4 ways

The one's place can be filled in = 3 ways

Therefore, by fundamental principle of multiplication the required number of 3 digit numbers = $4 \times 4 \times 3 = 48$

(i) If repetition of digits is allowed $\boxed{H}\boxed{T}\boxed{U}$

The hundred's place can be filled in = 4 ways

The ten's place can be filled in = 5 ways

The unit's place can be filled in = 5 ways

Therefore by fundamental principle of multiplication the required number of 3 digit numbers = $4 \times 5 \times 5 = 100$

Example 16: Find the number of different signals that can be generated by arranging at least 4 flags in order (one below the other) on a vertical staff, if 5 different flags are available.

Solution: A signal can consist of either 4 flags or 5 flags. Now, let us count the possible number of signals consisting of 4 flags and 5 flags separately and then add the respective numbers.

There will be as many 4 flag signals as there are ways of filling in 4 vacant place in succession by the 5 flags available.

By fundamental principle of multiplication, the number of ways is $5\times4\times3\times2 = 120$

Counting the same way the number of 5 flag signals is $5\times4\times3\times2\times1 = 120$

Therefore, the required number of signals = 120 + 120 = 240

Exercise 1.2

1. Find the number of 4 letter words, with or without meaning, which can be formed using the letters of the word HONEST, when the repetition of the letters is not allowed.

2. How many 3 digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated?

3. How many 4-letter code can be formed using the first 10 letters of the English alphabet, if

   (i) no letter is repeated

   (ii) repetition of letters is allowed

4. A tennis club consists of 8 boys and 11 girls. In how many ways can a mixed doubles team be chosen?

5. There are 5 vacant seats in a row. In how many ways can 3 men sit.

6. Find the total number of ways of answering 6 multiple choice questions, if each question has 4 choices.

7. Find the number of three digit even positive integers.

8. Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if 5 different flags are available.

9. A coin is tossed 4 times and the outcomes are recorded. How many different outcomes are possible?

10. There are 5 true-false questions in a test. If no two students have answered the same sequence of answers and no student has given all correct answers. How many students are there is the class for this to happen?

11. If each user on a computer system has a password which is eight characters long where each character is an upper case letter or a digit.

Each password must contain at least one digit. How many password are possible.

12. In a class test a teacher decides to give 5 questions one each from first five exercises of the textbook. If the first five exercises have 7, 12, 6, 10 and 3 questions respectively. Find the number of ways in which the question paper can be set.

13. How many numbers are there between 100 and 1000 such that 7 is in the units place.

14. How many numbers having 5 digits can be formed with the digits 0, 2, 3, 4 and 5 if repetition of digits is not allowed. How many of these are divisible by 5?

15. There are 21 towns in district connected by railways. Find the number of tickets required by the railways so that a passenger can travel from one town to another.

Answers:

1. 720
2. 50
3. (i) 5040

   (ii) 10000
4. 88
5. 60
6. 4096
7. 450
8. 320
9. 16
10. 31
11. $(36)^8 - (26)^8$
12. 15120

13. 90

14. 96, 42

15. 420

6.4 Permutations

Definition 1: A permutation is an arrangement of objects in a definite order taken some or all at a time.

The word permutation means arrangements. Each of the different arrangements that can be made by taking some or all of a given set of objects at a time is known as their permutation.

Let us understand this with the help of an example.

The postal department has to inscribe different codes on the consignment being shipped using digits 5, 7 and alphabet A. How many different codes can be generated using one or more character out of 5, 7 and A.

(i)	Codes using one character	:	5, 7, A
(ii)	Codes using two character	:	57, 75, 5A, A5, 7A, A7
(iii)	Codes using three character	:	57A, 5A7, 75A, 7A5, A57, A75

Now each of the above three gives us number of permutations of 3 characters (2 digits and 1 alphabet) taken one, two or all three at a time.

If we have to select two characters from 5, 7 and A. It can be done by writing 57, 5A, 7A. Thus the number of ways of selecting two characters is 3. Now each of these selections made above give rise to two arrangements i.e., 57 can be arranged in two ways as 57 and 75; similarly 5A can be arranged as 5A and A5; 7A as 7A and A7. Thus the total number of arrangement of 3 characters taken two at a time is 6, there are:

57, 75, 5A, A5, 7A, A7

Thus we have two types of classification (i) selection (ii) arrangement. Selections of objects are called combinations and arrangements of objects are called permutations.

Note: Permutation of objects taken some or all at a time, not only includes every combination of those objects but all other arrangements which each of these combination give rise to.

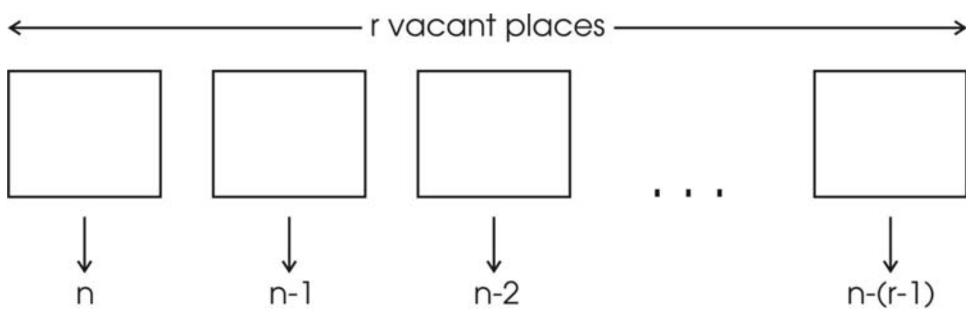
The number of permutations of n different things taken r at a time where repetition is not allowed is denoted by ^nP_r or P(n r) where n and r are positive integers and r $\leq$ n.

Permutations when all the objects are distinct.

Theorem 1: The number of permutations of n different objects taken r at a time, where o < r $\leq$ n and objects do not repeat is

$$^nP_r = n\,(n-1)\,(n-2).....\,(n-r+1) = \frac{n!}{(n-r)!}$$

Proof: There will be as many permutations as there are ways of filling r vacant places by n distinct objects.



The first place can be filled in n ways, following which the second place can be filled in (n–1) ways and then the third place can be filled in (n–2) ways.

Continuing so on the rth place can be filled in (n–(r–1)) ways.

Therefore, the number of ways of filling r vacant places in succession is n (n–1) (n–2) ..... (n–r+1)

= $n!$, if r = n

$$\frac{n\left(n-1\right)\left(n-2\right)\ldots\left(n-r+1\right)\left(n-r\right)\left(n-r-1\right)\ldots 3.\ 2.\ 1}{\left(n-r\right)\left(n-r-1\right)\ \ldots\ 3.2.1}, \text{if } o<r<n$$

This expression for ^nP_r on simplifying gives

$\Rightarrow\ {}^nP_r = \dfrac{n!}{(n-r)!}$, where 0 < r < n

Note : As 0! = 1

${}^nP_r = \dfrac{n!}{(n-r)!}$ stands true even for r=n

Counting permutations is merely counting the number of ways in which some or all objects at a time are arranged. Arranging no objects is same as leaving all objects behind and we knows this can be done in only one way. Hence, ${}^nP_o = 1$ makes sense. So ${}^nP_r = \dfrac{n!}{(n-r)!}$ stands true even for r = 0 Hence, ${}^nP_r = \dfrac{n!}{(n-r)!}$, $o \leq r \leq n$.

Theorem 2: The number of permutations of n different objects taken r at a time, when repetition is allowed, is n^r.

Proof: Since each of the r vacant places can be filled in n ways. Therefore the number of ways are

$$n \times n \times n \times \ldots. \times n = n^r$$

Example 17: Find the value of n such that

(i) $^nP_4 = 20\ {}^nP_2$, n > 3

(ii) $\dfrac{{}^nP_4}{{}^{n-1}P_4} = \dfrac{5}{3}$, n > 4

Solution:

(i) Given that ${}^nP_4 = 20\,{}^nP_2$

$$\Rightarrow \quad \frac{n!}{(n-4)!} = 20\,\frac{n!}{(n-2)!}$$

$$\Rightarrow \quad \frac{n!}{(n-4)!} = 20\,\frac{n!}{(n-2)(n-3)(n-4)!}$$

$\Rightarrow$ (n –2) (n –3) = 5 × 4 ($\therefore$ n > 3)

$\Rightarrow$ n – 2 = 5 or n – 3 = 4

$\Rightarrow$ n = 7

(ii) Given that $\frac{{}^nP_4}{{}^{n-1}P_4} = \frac{5}{3}$

$\Rightarrow$ $3\,{}^nP_4 = 5\,{}^{n-1}P_4$

$$\Rightarrow \quad 3\,\frac{n!}{(n-4)!} = 5\,\frac{(n-1)!}{(n-5)!}$$

$$\Rightarrow \quad \frac{3n(n-1)!}{(n-4)(n-5)!} = \frac{5(n-1)!}{(n-5)!}$$

$\Rightarrow$ 3n = 5 (n–4) (Cancelling common factors on both sides)

$\Rightarrow$ 3n = 5n – 20

$\Rightarrow$ 2n = 20 $\Rightarrow$ n = 10

Example 18: Find r if

(i) ${}^5P_r = 2\,{}^6P_{r-1}$

(ii) $5\,{}^4P_r = 6\,{}^5P_{r-1}$

Solution:

(i) We have ${}^5P_r = 2\,{}^6P_{r-1}$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = 2 \times \frac{6!}{(6-r+1)!}$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = 2 \times \frac{6.5!}{(7-r)(6-r)(5-r)!}$$

$\Rightarrow$ (7–r) (6–r) = 12

$\Rightarrow$ $r^2 - 10r - 3r + 30 = 0$

$\Rightarrow$ (r – 10) (r – 3) = 0

$\Rightarrow$ r = 10 or 3

$\Rightarrow$ r = 3 $(\because r \le 5)$

(ii) $5\,{}^4P_r = 6\,{}^5P_{r-1}$

$$\Rightarrow \quad 5 \times \frac{4!}{(4-r)!} = 6 \times \frac{5!}{(5-r+1)!}$$

$$\Rightarrow \quad \frac{5!}{(4-r)!} = \frac{6 \times 5!}{(6-r)(5-r)(4-r)!}$$

$\Rightarrow$ (6–r) (5–r) = 6

$\Rightarrow$ $r^2 - 11r - 24 = 0$

$\Rightarrow$ (r – 8) (r – 3) = 0

$\Rightarrow$ r = 8 or 3

$\Rightarrow$ r = 3 $(\because r \le 4)$

Example 19: How many 4-digit numbers can be formed using the digits 1 to 9 if no digit is repeated.

Solution: All possible 4-digit numbers are arrangements of 9 digits taken 4 at a time i.e., ${}^9P_4$

$${}^9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = 9 \times 8 \times 7 \times 6 = 3024$$

Example 20: How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once?

Solution: Number of arrangement of 8 distinct letters taken all at a time is

= $^{8}P_{8} = 8! = 40,320$

Example 21: How many 4-digit numbers with each digit even can be formed if:

(i) repetition of digits is not allowed

(ii) repetition of digits is allowed

Solution:

(i) Four digit even digit numbers if repetition is not allowed: $\boxed{Th}\boxed{H}\boxed{T}\boxed{U}$

Even digits are 0, 2, 4, 6 and 8.

The thousand's place can be filled in 4 ways ($\because$ 0 can not come)

The remaining 3 places (i.e. hundred's, ten's & unit)

Can be filled in $^{4}P_{3}$ ways ($\because$ 0 can come)

All possible 4 digit numbers = $4 \times {}^{4}P_{3}$

$$= 4\left[\frac{4!}{(4-3)!}\right] = 96$$

(ii) Four digit numbers with each digit even (with repetition):

The thousand place can be filled in 4 ways (except 0)

Then each of the remaining 3 digits can be filed in 5 ways

So all possible 4 digit numbers with all digits even and with repetition

$= 4 \times 5 \times 5 \times 5 = 500$

Example 22: In how many different ways can the letters of the word SUNDAY be arranged? How many of these begin with S? How many of these arrangements begin with S but does not end with Y.

Solution: The word SUNDAY contains 6 distinct letters. So the number of different ways are arrangement of the 6 letters taken all at a time = ${}^{6}P_{6} = 6! = 720$

The arrangements that begin with S can be obtained by keeping S fixed in the first position and then arranging the remaining 5 letters. This can be done in ${}^{5}P_{5} = \underline{\lfloor 5} = 120$ ways.

Also the arrangements that begin with S and end with Y can be obtained by keeping S and Y fixed in the first and last position respectively and arranging the remaining 4 letters in ${}^{4}P_{4} = 4! = 24$ ways.

Number of arrangements that begin with S but does not end with Y

= (Number of arrangements that begin with S) – (Number of arrangements that begin with S and end with Y)

$$= {}^{5}P_{5} - {}^{4}P_{4} = \underline{\lfloor 5} - \underline{\lfloor 4} = 96$$

Example 23: In how many ways can a group of 8 friends having different heights stand in a row for a group photograph if

(i) they stand in ascending order of their heights (from left to right)

(ii) the tallest and the shortest should not stand together.

Solution:

(i) If the students stand in ascending order of their heights from left to right then each of them can stand in only one position according to their height.

So number of ways = $1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 = 1$ way

(ii) Number of ways in which the tallest and the shortest friend would stand together can be obtained by considering those two persons as a single unit. Then we need to arrange 7 in all which can be done in ${}^{7}P_{7} = 7!$ ways.

Now the tallest and the shortest which come together can be arranged among themselves in 2! ways.

So the total number of arrangements when the tallest and the shortest are together is $7! \times 2!$

Number of arrangements when the tallest and the shortest are not together

= (All possible arrangements of 8 friends) – (Arrangements when tallest and shortest are together)

$= 8! - 7! \times 2!$

$= 8.7! - 7! \times 2$

$= 7!\,(8-2)$

$= 6 \times 7!$

Example 24: Find the number of arrangements of the letters of the word FRAGILE. In how many of these arrangements

(i) do all the vowels occur together.

(ii) do all the vowels occur together and all the consonants occur together.

(iii) do all the vowels never occur together.

(iv) do the vowels occupy only even places.

(v) do the vowels occupy only odd places.

Solution: Number of arrangements of the letters of the word FRAGILE = ${}^7P_7 = 7!$

(i) There are 7 letters in the word FRAGILE, in which there are 3 vowels, namely A, I and E. Since the vowels have to occur together, we can assume them as single object (AIE). This single object along with 4 remaining letters (objects) will be counted as 5 objects.

Permutation of these 5 objects taken all at a time = ${}^5P_5 = 5!$

But A, I, E can be arranged among themselves in = ${}^3P_3 = 3!$

Hence by multiplication principle number of permutation

$= 5! \times 3! = 120 \times 6 = 720$

(ii) One of the arrangement of all vowels coming together and all consonants coming together is (AIE) (FRGL)

Considering (AIE) to be one group and (FRGL) to be another group the two objects can be arranged in ${}^2P_2 = 2!$ ways

The vowels A, I, E can be arranged among themselves in = ${}^3P_3 = 3!$ ways

The consonants F, R, G, L can be arranged among themselves in = ${}^4P_4 = 4!$ ways

So by multiplication rule the total number of arrangements are = $2! \times 3! \times 4!$

$= 2 \times 6 \times 24 = 288$ ways

(iii) All vowels never occur together = (All the arrangements of the word FRAGILE) – (the arrangements with all vowels together)

$= 7! - 5! \times 3!$

$= 7 \times 6 \times 5! - 5! \times 6$

$= 5!\,(42 - 6)$

$= 120 \times 36 = 4320$

(iv) The word FRAGILE consists of 7 letters of which 3 are vowels and 4 are consonants.

1	2	3	4	5	6	7

The vowels can occupy even places if they occur at second, fourth and sixth positions.

The three vowels can occupy the three positions in ${}^3P_3 = 3!$ ways

The remaining four positions can be filled up by 4 consonants in ${}^4P_4 = 4!$ ways

So the required number of ways = $3! \times 4! = 144$

(v) The three vowels can occupy the 4 odd places in ${}^4P_3$ ways. The remaining 4 places can be filled up by 4 consonants in ${}^4P_4$ ways.

So the required arrangements = ${}^4P_3 \times {}^4P_4 = \frac{4!}{(4-3)!} \times 4!$

$= 4! \times 4! = 24 \times 24 = 576$

Exercise 1.3

1. Find n if ${}^{n-1}P_3 : {}^nP_4 = 1 : 9$

2. Find r if

 (i) ${}^9P_r = 3024$

 (ii) ${}^5P_r = 2\ {}^6P_{r-1}$

3. Prove the following:

 (i) ${}^nP_n = 2 \,.\, {}^nP_{n-2}$

 (ii) ${}^{n-1}P_r + r \,.\, {}^{n-1}P_{r-1} = {}^nP_r$

4. How many 3-digit numbers are there with no digit repeated?

5. How many 4-digit even numbers can be formed using the digits 1, 2, 3, 5, 7 and 8 if repetition of digits is not allowed?

6. How many numbers between 6000 and 7000 formed with the digits 0, 1, 5, 6, 7 and 9 are divisible by 5 if

 (i) repetition of digits is allowed

 (ii) repetition of digits is not allowed

7. A family of 6 brothers and 4 sisters is to be arranged for a photograph in one row. In how many ways can they be seated so that

 (i) all the sisters sit together

 (ii) no two sisters sit together

8. How many words, with or without meaning can be made from the letters of the word TUESDAY, assuming that no letter is repeated, if

   (i) all letters are used at a time

   (ii) 5 letters are used at a time

   (iii) all letters are used but first and last letter is a vowel

9. In how many ways can the letters of the word PERMUTATIONS be arranged if the

   (i) words start with P and end with S

   (ii) there are 5 letters between P and S

   (iii) vowels are all together

10. How many words with or without meaning can be formed using all the letters of the word LAUGHTER if

   (i) the words start with L but does not end with R

   (ii) no two vowels come together

   (iii) the relative positions of vowels and consonants remains unchanged

11. In how many ways can 5 Mathematics, 4 English and 3 Accountancy books can be arranged in a shelf if

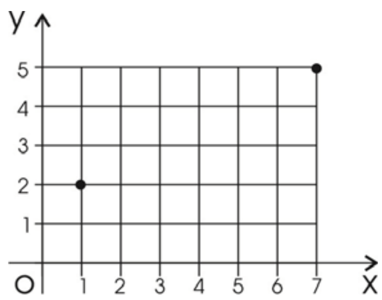
   (i) all books on the same subject are together

   (ii) No two books on the same subject are together

12. Find the rank of the word LATE, if the letters of the word LATE are permuted and words so formed are arranged as in dictionary.

13. Find the numbers of words with or without meaning which can be made using all the letters of the word AGAIN. If all these words are arranged as in dictionary what will be the 49^{th} word, 50^{th} word?

14. Determine the number of paths in the xy–plane from (1, 2) to (7, 5), where each such path is made up of individual steps going one unit to the right (R) or one unit upwards (U)



15. How many positive integers greater than 5,000,000 can be formed using the digits 2, 3, 3, 5, 5, 6, 8?

16. The board of directors of a pharmaceutical company has 10 members. An upcoming stockholder's meeting is scheduled to approve a new president, vice president, secretary and a treasurer. How many different ways the four can be appointed.

17. In how many ways can 9 people be arranged around a circular table if two people insist on sitting next to each other.

18. If the letters of the word ADINI are arranged as in dictionary, then what is the 46^{th} word?

19. If the letters of the word SCHOOL are arranged as in dictionary, then find the rank of the word SCHOOL.

Answers:

1. $n = 9$

2. (i) $r = 4$ (ii) $r = 3$

4. 648

5. 120

6. (i) 71 (ii) 24

7. (i) $\lfloor 7 \times \lfloor 4$ = 120960

(ii) $\lfloor 6 \times {}^{7}P_{4}$ = 604800

8. (i) 5040 (ii) 2520 (iii) 720

9. (i) 1814400 (ii) 21,772,800 (iii) 2419200

10. (i) 4320 (ii) $5! \times {}^{6}P_{3} = 14400$ (iii) $3! \times 5! = 6 \times 120 = 720$

11. (i) 103680 (ii) ${}^{5}P_{5} \times {}^{6}P_{4} \times {}^{10}P_{3} = 31104000$

12. 14

13. NAAGI & NAAIG

14. 84

15. 720

16. 5040

17. $7! \times 2$ = 10,080

18. INDIA

19. 303

We can state without proof the following theorems:

Theorem 1: The number of permutations of n objects, where p objects are of the same kind and rest, if any, are all different = $\frac{n!}{p!}$

We have a more general theorem.

Theorem 2: The number of permutations of n objects, where p_1 objects are of one kind, p_2 objects are of second kind, ..., p_k objects are of k^{th} kind and rest if any, are all distinct is $\frac{n!}{p_1!\,p_2!...p_k!}$.

Example 25: Find the number of permutations of the letters of the following words.

(i) COMMERCE

(ii) MATHEMATICS

Solution:

(i) The word COMMERCE has 8 letters, in which C appears twice, M appears twice, E appears twice and the remaining letters only once.

So total number of arrangements = $\frac{8!}{2!\,2!\,2!}$ = 5040

(ii) The word MATHEMATICS has 11 letters, in which A, M and T occurs twice and remaining letters are distinct.

So total number of arrangements = $\frac{11!}{2!\,2!\,2!}$ = 4989600

Example 26: Find the number of permutations of the letters of the word ENGINEERING. In how many of these arrangements:

(i) do the words begin with E and end with G

(ii) do all the vowels come together

(iii) do all the vowels never come together

(iv) no two vowels come together

Solution: The word ENGINEERING has 11 letters. Writing all identical letters together we get ÉEEEIINNNGGR'.

It contains 3E's, 2I's, 3N's, 2G's and 1R

So the number of arrangements = $\frac{11!}{3!\,2!\,3!\,2!}$ = 277200

(i) If we fix the first and last letter as E and G the remaining 9 letters contains 2E's, 3N's, 2I's, 1G and 1R.

So the number of arrangements = $\frac{9!}{3!\,2!\,2!}$ = 15,120

(ii) Let us consider all vowels as one letter (i.e. object). So in all we have 7 letters in which we have 3N's, 2G's, 1R and combination of vowels EEEII as 1 object.

The number of ways these 7 letters can be arranged among themselves in $= \frac{7!}{3!\,2!}$ ways and for each of these arrangements the 5 vowels can be arranged among themselves $= \frac{5!}{2!\,3!}$ ways.

So all vowels can come together in $= \frac{7!}{3!\,2!} \times \frac{5!}{2!\,3!} = 4200$

(iii) All vowels not coming together = All arrangements – Arrangements when all vowels come together

$= 277200 - 4200$

(iv) The word ENGINEERING consists of 5 vowels EEEII and 6 consonants NNNGGR.

Let us arrange the 6 consonants first. Since it contains 3N's, 2G's and IR, therefore the number of arrangements are $\frac{6!}{3!\,2!} = 60$ ways.

$* \, C \, * \, C \, * \, C \, * \, C \, * \, C \, * \, C \, *$

Now once the consonants 'C' are arranged the vowels EEEII can be placed in any of the 5 positions out of 7 shown by * symbol in ${}^7C_5$ ways. Further the vowels can be arranged among themselves in $\frac{5!}{3!\,2!}$ ways.

So the number of ways in which vowels can be arranged is

$${}^7C_5 \times \frac{5!}{3!\,2!} = 21 \times 10 = 210 \text{ ways}$$

By multiplication rule the arrangements of the word ENGINEERING in which all vowels come together is

$60 \times 210 = 12600$

Example 27: In how many ways can the letters of the word ASSASSINATION be arranged? In how many of these arrangements the four S's do not come together.

Solution: The word ASSASSINATION has 13 letters. It contains 3A's, 4S's, 2I's, 2N's, 1T and 1O.

So the number of arrangements = $\frac{(13)!}{3!\,4!\,2!\,2!}$

$$= \frac{(13)!\,12\times11\times10\times9\times8\times7\times6\times5\times4!}{3!\,4!\,2!\,2!}$$

= 10810800

To find the number of arrangement if 4S's come together, we consider the four S's to be one letter. Then we shall have 10 letters in all with 3A's, 2I's, 2N's, 1T, 1 O and 1 combined S.

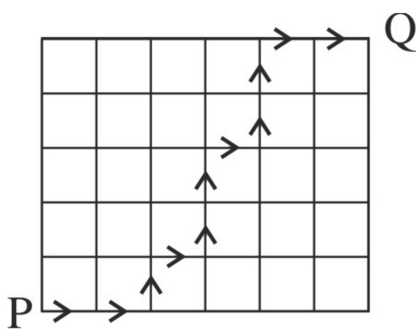
So number of arrangements when 4S's come together is $\frac{10!}{3!\,2!\,2!}$ = 151200

Hence number of arrangements when four S's do not come together is

= 10810800 – 151200 = 10659600

Example 28: A boy is to walk from P to Q. However, he can take a right step or an upward step, but not necessarily in the order shown in given figure. Find the number of possible paths he can take.

Solution: In all paths from P to Q the boy has to take a total of 11 steps five upward and six right steps. So number of paths is arrangement of 11 steps $\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\uparrow\uparrow\uparrow\uparrow\uparrow$



Which can be done in $\frac{11!}{5!\,6!} = \frac{11\times10\times9\times8\times7}{5\times4\times3\times2\times1}$ = 462

Example 29: The MASSASAUGA is a brown and white venomous snake found in North America. How many arrangements can be made from this word. Find the number of arrangements if all vowels are together.

Solution: The word 'MASSASAUGA' has 10 letters. It contains 4A's, 3S's and one each of M, U and G.

So the number of arrangements are $\frac{10!}{4!\,3!}$ = 25200

To find number of arrangements when all vowels are together we consider 4A's and IU i.e., AAAAU as one letter.

So now there are 6 letters having 3S's, 1M, 1G, and 1 combined vowels, which can be arranged in $\frac{6!}{3!}$

But the vowels AAAAU can be arranged among themselves in $\frac{5!}{4!}$ ways.

Hence number of arrangements when all vowels come together is $\frac{6!}{3!} \times \frac{5!}{4!}$ = 600

Example 30: How many arrangements of the letters of word SOCIOLOGICAL are there if A and G are adjacent.

Solution: Since A and G are adjacent, therefore we consider them as one unit
(letter). The word SOCIOLOGICAL has 11 letters in which 3O's, 2C's, 2I's, 2L's, 1S and 1 combined GA.

Hence number of arrangements are $\frac{11!}{3!\,2!\,2!\,2!}$

But GA can also be written as AG if they are adjacent.

Therefore number of arrangements of SOCIOLOGICAL if A and G are adjacent is $2! \times \frac{11!}{3!\,2!\,2!\,2!} = 1663200$

6.5 Circular Permutation

Till now we have done linear permutation i.e., arrangement in a line. In case the objects are arranged in the form of a circle it is known as circular permutation. Since a circle does not have any extremity therefore the number of arrangements in a circle depends on the relative position of the objects.

Theorem 1: The number of permutations of n objects in a circle taken all at a time is (n–1)!

Arrangements of beads to form a garland is also a circular permutation, but a garland can be turned upside down. So the two orders of arrangements i.e. clockwise and anticlockwise in not distinguishable, further reducing the number of arrangements to half. So in arrangements of beads in a necklace or arrangements of flowers in a garland we have the following therem.

Theorem 2: If clockwise and anticlockwise order of arrangements are not distinguishable,

Then number of permutations of n distinct objects is $\frac{(n-1)!}{2}$

Example 31: In how many ways can 7 students be made to sit in a (i) line (ii) circle.

Solution:

(i) The number of ways in which 7 students can be made to sit in a line = $^7P_7$ = 7! = 5040

(ii) The number of ways in which 7 students can be made to sit in a circle = (7–1)! = 6! = 720

Example 32: In how many ways can 4 men and 4 women be seated at a round table if

(i) no restriction is imposed

(ii) two particular women must sit together

(iii) all women must sit together

Solution:

(i) Total number of persons = 4 + 4 = 8

Number of ways in which 8 persons can be seated at a round table

= (8 – 1)! = 7! = 5040

(ii) Since the two particular women are to sit together we consider them as one. Now total number of arrangements of 7 persons at a round table is (7–1)! = 6!

The two women who are sitting together can be arranged among themselves in 2! ways.

So the total number of arrangements = 2! × 6! = 1440

(iii) The number of arrangements in which all the women would sit together can be obtained by considering all 4 women to be single entity. Then there will be total of (4 + 1) i.e. 5 persons which can be selated in 4! ways. Now the 4 women who are sitting together can be arranged among themselves in 4! ways.

So the total number of arrangements at a round table if the 4 women sit together is 4! × 4!

= 24 × 24 = 576 ways.

Example 33: 17 persons are invited to a party. How many seating arrangements are possible at a round table if

(i) the host can sit any where

(ii) the bigger chair is fixed for the host

(iii) there are two pairs of indistinguishable twins among the guests.

Solution:

(i) Number of ways in which 18 persons (17 invitees and 1 host) can be seated at a round table without any restriction is (18–1)! = 17!

(ii) The bigger chair is fixed for the host, the remaining 17 persons can be seated at a round table in 17! ways.

(iii) 18 persons can sit at a round table in 17! ways. If there are 2 pairs of indistinguishable twins, than the number of arrangements are $\frac{17!}{2!\,2!}$.

Example 34: Find the number of ways in which 8 different beads can be arranged to form a necklace.

Solution: Eight different beads can be arranged in a circle in (8–1)! = 7! ways. Since there is no distinction between clockwise and anticlockwise arrangements, so the required number of arrangements are $\frac{7!}{2}$.

Example 35: How many necklaces can be made using 20 beads, 8 being blue, 5 green, 5 yellow and 2 red.

Solution: We known 20 distinct beads can be arranged in 19! ways. But since 8 of them are blue, 5 green, 5 yellow and 2 red therefore the number of arrangements are $\frac{19!}{8!\,5!\,5!\,2!}$.

Now, since clockwise and anticlockwise arrangements are identical, therefore to total number of arrangements are $\frac{1}{2}\left(\frac{19!}{8!\,5!\,5!\,2!}\right)$.

Example 36: Find the number of words with or without meaning which can be made using all the letters of the word MOTHER. If these words are written as a dictionary find the rank of the word mother.

Solution: There are 6 letters in the word MOTHER, all of which are distinct. So the required number of words is ${}^{6}P_{6}$ = 6! = 720. The letters of the word MOTHER in alphabetical order are E, H, M, O, R and T.

To obtain the number of words starting with E, we fix the letter E at the extreme left position, we then rearrange the remaining 5 letters taken all at a time in ${}^{5}P_{5}$ = = 5! ways. The same number of words we will obtain if we start with H. So before the first letter starting with M, we have $2 \times 5! = 240$ words.

Now considering the first two letters of the words starting with M, we get ME, MH before MO.

So to obtain the number of words starting with ME we fix the first two letters ME and arrange the remaining 4 letter taken all at a time in ${}^{4}P_{4}$ = 4! ways. The same number of words we will obtain with MH. So before MO, the number of words starting with ME and MH are $2 \times 4! = 48$ words.

Now the sequence of first three letters before MOT will be MOE, MOH and MOR, each one of these when fixed will give ${}^{3}P_{3}$ = 3! words. Thus the number of words starting with MOE, MOH and MOR are $3 \times 3! = 18$. Next, the first 3 words starting with MOT are MOTEHR, MOTERH & MOTHER.

Thus rank of word MOTHER = 240 + 48 + 18 + 3 = 309

6.5 Permutations with Restrictions

A restricted permutation is one in which certain objects are always included or excluded. It also includes those permutations in which some objects have to be placed in designated place or certain objects come together.

Let us state some of these theorems.

Theorem 1: The number of permutations of n different things taken r at a time in which m particular things never occur is obtained by arranging r objects out of the remaining n – m objects in ${}^{n-m}P_{r}$ ways, where $n - m \geq r$.

Example 37: Find the number of ways of selecting a president, secretary and a treasurer from a board of 8 members, if 2 members who were holding one of these posts earlier cannot be nominated again.

Solution: Since 2 members are ruled out, so number of ways of selecting a president, secretary and a treasurer from among 6 members is number of arrangements of 6 members taken 3 at a time i.e., $^6P_3 = \frac{6!}{(6-3)!} = 120$.

Theorem 2: The number of permutations of n different things taken r at a time in which m particular things always occur is $^rP_m \times {}^{n-m}P_{r-m}$ ways.

Example 38: Find how many 4 digit numbers with distinct digits can be formed using the digits 1, 2, 3, 4, 5, 7 and 9 if each number contains two even digits.

Solution: Since 2 and 4 are the only two even digits given, so all number should contain them. So these two even digits can be placed in a 4 digit number in $^4P_2$ ways.

Now once these even digits are placed, the remaining 2 digits can be arranged in $^5P_2$ ways. So the number of 4 digit numbers are:

$$^4P_2 \times {}^5P_2$$

$$= \frac{4!}{(4-2)!} \times \frac{5!}{(5-2)!} = \frac{4!}{2!} \times \frac{5!}{3!}$$

$$= 3 \times 4 \times 4 \times 5 = 240$$

6.6 Combination

A school library offers two books for an academic year from a list of 12 books. If you have to put a right mark √ against a pair of books you desire in a list mentioning all possible pairs of books. Can you think of the number of choices you have to choose from.

What difference the order will make if you borrow the same two books. Give it a thought! Clearly, in this case the order is not important. Such an arrangement of objects in which order makes no difference is called a combination.

Let us consider some more illustrations.

On the first day of a class each of the 25 students shook hands with each other. How many handshakes took place. X shaking hands with Y and Y with X are not two different handshakes. Here again order is not important. There will be as many handshakes as there are combinations of 25 different students taken 2 at a time.

Now, we obtain the formula for finding the number of combinations of n different objects taken r at a time, denoted by ${}^nC_r$.

Let us assume there are 4 students A, B, C, D of a school who actively participates in various quiz competitions. A team of two students is to be selected to represent the school in an inter school quiz contest. In how many ways can we do so?

In fact the various possibilities of selecting the 2 students of the team are AB, AC, AD, BC, BD and CD. Here AB and BA are the same combination as order does not alter the combination. This is the reason we have not included BA and likewise some other arrangements in the list.

Here each selection is a combination of 4 different objects taken 2 at a time. There are as many as 6 combinations of 4 different objects taken 2 at a time, i.e., ${}^4C_2 = 6$.

Corresponding to each combination in the list, we can arrive at 2! permutations as 2 objects in each combination can be arranged in 2! ways.

Hence the number of permutations = ${}^4C_2 \times 2!$

On the other hand, the number of permutations of 4 different things taken 2 at a time = ${}^4P_2$

Therefore ${}^4P_2 = {}^4C_2 \times 2! \quad \Rightarrow \quad {}^4C_2 = \dfrac{{}^4P_2}{2!}$

or $\quad {}^4C_2 = \dfrac{4!}{(4-2)!\,2!}$

Similarly, if we have 5 different objects and we have to make combinations taking 3 at a time, we can show

$$^5C_3 = \frac{5!}{(5-3)!\,3!}$$

These examples suggest the following relationship between permutation and combination.

Remark: $^nP_r = {^nC_r}\ r!$, where $0 < r \leq n$

Corresponding to each combination in ^nC_r, we have r! permutations, because r objects in every combination can be arranged in r! ways.

Therefore the total number of permutations of n different things taken r at a time is $^nC_r \times r!$. On the other hand we know it is ^nP_r.

Thus, $^nP_r = {^nC_r} \times r!$, $0 < r \leq n$

This result gives us an important theorem of combination.

Theorem: The number of combinations of n different objects taken r at a time is given by

$$^nC_r = \frac{n!}{r!(n-r)!}$$

Note 1: It can be easily seen that

$$^nC_0 = 1 = {^nC_n}$$

$$^nC_0 = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1 \qquad (\because 0! = 1)$$

$$^nC_n = \frac{n!}{n!(n-n)!} = \frac{n!}{n!} = 1 \qquad (\because 0! = 1)$$

Moreover, ^nC_n means the number of ways of selecting n objects from n distinct objects i.e. the whole group which can be done in 1 way.

Also $^nC_0$ means selecting nothing at all and it is equivalent to leaving behind all the objects and we know there is only one way of doing so.

Hence $^nC_0 = 1$

Note 2: ${}^{n}C_{n-r} = \dfrac{n!}{(n-r)!\,(n-(n-r))!} = \dfrac{n!}{(n-r)!\,r!} = {}^{n}C_{r}$

Thus, ${}^{n}C_{r} = {}^{n}C_{n-r}$

This formula can be logically explained like this. The number of combination of n distinct objects taken r at a time is selecting a group of r objects out of n. Each time n objects are selected, we leave behind the remaining (n – r) objects. So number of combination of n objects taken r at a time is equal to number of combination of n objects taken (n – r) at a time.

Note 3: We know ${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$

$$\Rightarrow \quad {}^{n}C_{r} = \frac{n(n-1)(n-2).....(n-r+1)(n-r)!}{r!(n-r)!}$$

$$\Rightarrow \quad {}^{n}C_{r} = \frac{n(n-1)(n-2).....(n-r+1)}{r!}$$

Thus, ${}^{5}C_{2} = \dfrac{5\times 4}{2\times 1} = 10$, ${}^{51}C_{3} = \dfrac{51\times 50\times 49}{3\times 2\times 1} = 20825$

Note 4: If ${}^{n}C_{a} = {}^{n}C_{b} \Rightarrow$ Either a = b or a = n – b i.e., n = a + b

Theorem: ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$

Proof: ${}^{n}C_{r} + {}^{n}C_{r-1}$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!}{r(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!\,(n-r)!} \times \frac{n-r+1+r}{r(n-r+1)}$$

$$= \frac{(n+1)n!}{r(r-1)!(n-r+1)(n-r)!} = \frac{(n+1)!}{r!(n-r+1)!}$$

$= {}^{n+1}C_r$

Example 39: If $n \times {}^{19}C_4 = {}^{19}P_4$, find n.

Solution: $n \times {}^{19}C_4 = {}^{19}C_4 \times 4!$

$\Rightarrow n = 24$

Example 40: Verify that ${}^{10}C_5 + {}^{10}C_6 = {}^{11}C_6$

Solution: LHS = ${}^{10}C_5 + {}^{10}C_6$

$$= \frac{10!}{5!\,5!} + \frac{10!}{6!\,4!}$$

$$= \frac{10!}{5! \times 5 \times 4!} + \frac{10!}{6 \times 5! \times 4!}$$

$$= \frac{10!}{5! \times 4!}\left[\frac{1}{5} + \frac{1}{6}\right]$$

$$= \frac{10!}{5! \times 4!} \times \frac{11}{5 \times 6}$$

$$= \frac{11 \times 10!}{6 \times 5! \times 5 \times 4!} = \frac{11!}{6! \times 5!} = {}^{11}C_6 = \text{RHS}$$

Hence verified

Example 41: Evaluate ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7$

Solution: $({}^{15}C_8 + {}^{15}C_9) - ({}^{15}C_6 + {}^{15}C_7)$

$= {}^{16}C_9 - {}^{16}C_7$ $\qquad (\because {}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r)$

$= {}^{16}C_7 - {}^{16}C_7$ $\qquad (\because {}^{16}C_9 = {}^{16}C_{16-9})$

$= 0$

Example 42: Prove that $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

Solution: LHS = $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n!}{r!(n-r4)!} \times \frac{(r-1)!(n-r+1)!}{n!}$

$= \frac{(r-1)!}{r(r-1)!} \frac{(n-r+1)(n-r)!}{(n-r)!} = \frac{n-r+1}{r}$ = RHS

Example 43: If ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in AP, find n

Solution: Since, ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in AP

Therefore, ${}^nC_4 + {}^nC_6 = 2({}^nC_5)$

$= \frac{n!}{4!(n-4)!} + \frac{n!}{6!(n-6)!} = 2\left[\frac{n!}{5!(n-5)!}\right]$

$= \frac{1}{(n-4)(n-5)} + \frac{1}{30} = \frac{2}{5(n-5)}$

$= \frac{1}{30} = \frac{1}{n-5}\left[\frac{2}{5} - \frac{1}{n-4}\right]$

$= \frac{1}{6} = \frac{2n-13}{n^2-9n+20}$

$= n^2 - 21n + 98 = 0$

$= (n-7)(n-14) = 0$

$= n = 7$ or 14

Example 44: If (i) ${}^nC_4 = {}^nC_7$, find n

(ii) ${}^{11}C_r = {}^{11}C_{r+3}$, find r

Solution:

(i) ${}^nC_4 = {}^nC_7$

$= {}^nC_{n-4} = {}^nC_7$ $\quad (\because {}^nC_r = {}^nC_{n-r})$

$= n - 4 = 7$

$= n = 11$

(ii) $\quad {}^{11}C_r = {}^{11}C_{r+3}$

$\Rightarrow 11 = r + (r + 3) \quad (\because r \neq r+3$, for non-negative integer r)

$\Rightarrow 2r = 8$

$\Rightarrow r = 4$

Example 45: In how many ways can one select a cricket team of eleven from 17 players? If only 7 players can bowl and the team must include exactly 5 bowlers then how many different combinations are possible?

Solution: Number of combinations of 17 players taken 11 at a time is ${}^{17}C_{11} = {}^{17}C_6$

$$= \frac{17.16.15.14.13.12}{6.5.4.3.2.1} = 12376$$

Further the team has 7 bowlers out of which we have to choose 5 and 10 other players out of which we have to choose 6, which can be done in

${}^{7}C_5 \times {}^{10}C_6$ ways

$= {}^{7}C_2 \times {}^{10}C_4$

$$= \frac{7\times6}{2\times1} \times \frac{10\times9\times8\times7}{4\times3\times2\times1}$$

$= 4410$

Example 46: If there are seven distinct points on the circumference of a circle. Find

(i) How many chords can be drawn joining the points in all possible ways.

(ii) How many triangles can be drawn using any 3 of these 7 points as vertices.

(iii) How many quadrilaterals can be drawn joining any 4 points on the circle.

Solution:

(i) Number of chords = ${}^7C_2 = \frac{7\times6}{2\times1}$ = 21

(ii) Number of triangles = ${}^7C_3 = \frac{7\times6\times5}{3\times2\times1}$ = 35

(iii) Number of quadrilaterals = ${}^7C_4$ = 35

Example 47: To attend a joint summit, two countries send 5 delegates each to a negotiating table. A rectangular table is used with 5 chairs on each side. If each country is assigned a long side of the table, how much seating arrangements are possible.

Solution: No. of arrangements = 5! × 5! × 2

= 120 × 120 × 2 = 28800

Example 48: There are 9 points in a plane of which only 5 are collinear. Find the number of

(i) straight lines that can be formed joining two points.

(ii) triangles that can be formed joining any three points.

Solution:

(i) A straight line is formed by combination of 2 points and combination of 9 points taken 2 at a time is ${}^9C_2$. But the 5 collinear points will contribute only one line among themselves.

Hence the number of straight line = ${}^9C_2 - {}^5C_2 + 1$

$= \frac{9\times8}{2\times1} - \frac{5\times4}{2\times1} + 1 = 27$

Hence 27 lines can be drawn

(ii) We know a triangle is formed by combination of 3 non collinear points and combination of 9 points taken 3 at a time is ${}^9C_3$. But 5 collinear points among themselves will not form any triangle as all 3 points in them will be collinear.

Hence number of triangle = ${}^{9}C_3 - {}^{5}C_3 = {}^{9}C_3 - {}^{5}C_2$

$$= \frac{9\times8\times7}{3\times2\times1} = \frac{5\times4}{2\times1} = 84 - 10 = 74$$

Hence the number of triangles are 74

Example 49: What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these:

(i) four cards are of the same suit

(ii) four cards belong to different suits

(iii) three cards are of same colour and one different.

Solution: There will be as many ways of choosing 4 cards from 52 cards as there are combinations of 52 different objects, taken 4 at a time.

Therefore the required number of ways = ${}^{52}C_4$

$$= \frac{52!}{4!\,48!} = \frac{49\times50\times51\times52}{2\times3\times4} = 270725$$

(i) There are four suits: diamond, club, spade, heart and there are 13 cards of each suit. Therefore, there are ${}^{13}C_4$ ways of choosing 4 diamonds. Similarly, there are ${}^{13}C_4$ ways of choosing 4 clubs and ${}^{13}C_4$ ways each of choosing 4 spades and similarly 4 hearts.

Therefore the required number of ways

$$= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 = 4\times\frac{13!}{4!\,9!} = 2860$$

(ii) There are 13 cards in each suit. So. There are ${}^{13}C_1$ ways at choosing 1 card from 13 cards of each suit. By multiplication rule, the no of ways at selecting 4 cards belong to four different suits is ${}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = (13)^4$.

(iii) Since, there are only two colours i.e., red and black so there are two possibilities i.e., 3 red and 1 black or 3 black and 1 red. We know there are 26 red and 26 black cards.

Hence the number of ways = ${}^{26}C_3 \times {}^{26}C_1 + {}^{26}C_1 \times {}^{26}C_3$

$$= 2\left[\frac{26 \times 25 \times 24}{3 \times 2 \times 1} \times 26\right] = 135200$$

Example 50: A committee of 5 is to be selected from amongst 6 gentlemen and 5 ladies. Determine the number of ways if it is to contain at least 1 gentleman and 1 lady.

Solution: If the committee is to contain at least one gentleman and 1 lady, the various possibilities are:

1G and 4L or 2G and 3L or 3G and 2L or 4G and 1L where G stands for gentleman and L stands for an lady.

The required number of ways are:

$${}^6C_1 \times {}^5C_4 + {}^6C_2 \times {}^5C_3 + {}^6C_3 \times {}^5C_2 + {}^6C_4 \times {}^5C_1$$

$$= {}^6C_1 \times {}^5C_1 + {}^6C_2 \times {}^5C_2 + {}^6C_3 \times {}^5C_2 + {}^6C_2 \times {}^5C_1$$

$$= 6 \times 5 + \frac{6 \times 5}{2 \times 1} \times \frac{5 \times 4}{2 \times 1} + \frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times \frac{5 \times 4}{2 \times 1} + \frac{6 \times 5}{2 \times 1} \times 5$$

$$= 30 + 150 + 200 + 75$$

$$= 455$$

Exercise 1.4

1. If ${}^{18}C_r = {}^{18}C_{r+2}$, find ${}^rC_5$

2. If ${}^nC_r : {}^nC_{r+1} = 1:2$ and ${}^nC_{r+1} : {}^nC_{r+2} = 2:3$, find n and r

3. Show that, ${}^nC_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + ... + {}^{n+r}C_r = {}^{n+r+1}C_r$ (Hint: Write ${}^nC_0 = {}^{n+1}C_0$)

4. How many chords can be drawn through 17 points on a circle?

5. The number of diagonals of a polygon is twice the number of its sides. Find the number of sides of the polygon.

6. A box contains 6 red and 7 white balls. Determine the number of ways in which 4 red and 3 white balls can be selected.

7. In how many ways can a committee of 5 is to be formed from 4 teachers and 6 students so as to include at least 2 students.

8. A cricket team of 11 players is to be formed from 15 players. In how many different ways the team can be selected if

   (i) Two players who scored maximum runs and took maximum wickets respectively must be included.

   (ii) One who is not in form should be excluded.

9. In how many ways can a student choose a programme of 5 courses if 10 courses are available and 2 language courses are compulsory for every student.

10. In how many ways can 7 plus (+) signs and 5 minus (–) signs be arranged in a row so that no two (–) signs are together.

11. Twenty points no four of which are coplanar are in space. How many triangles do they determine? How many planes? How many tetrahedrons?

Answers:

1. r = 56

2. r = 4, n =14

4. 136

5. 7

6. 525

7. 246

8. (i) 715 (ii) 364

9. 56

10. 56

11. 1140, 1140, 4845

6.7 Combination with Repetition

We have seen number of arrangements of n distinct objects taken r at a time with repetition in n^r. Do we have situations where combinations of objects with repetition is a possibility.

Let us consider a practical example. During lunch break 5 friends go to school canteen where each one of them has one of the following: a burger, cold coffee or noodles. How many different purchases are possible? Let b, c and n represent a burger, cold coffee and noodles respectively.

Here we are concerned with how many of each items are purchased, not with the order in which they are purchased.

So the problem is one of selections or combinations with repetition. Some possible combinations are b b b c c b c c n n c c c c c etc.

Now we give an important theorem.

Theorem: Number of combination of r objects from a set with n objects when repetition of elements is allowed = ${}^{n+r-1}C_r$

In the problem discussed above, we have

n = 3 (possible food or drink)

r = 5 (friends)

Note: r > n is possible here

Number of combinations = ${}^{n+r-1}C_r$

$= {}^{3+5-1}C_5$

$$= {}^{7}C_{5} = \frac{7\times 6}{2\times 1} = 21$$

Example 51: How many ways one can select three gift vouchers from a box containing Rs. 100, Rs. 500, Rs. 1000 and Rs. 2000 gift vouchers? Assume that the order in which the vouchers are chosen does not matter, the vouchers of each denomination are identical and there are at least three gift vouchers of each type.

Solution: Since the order in which the vouchers are selected does not matter and vouchers of 4 different amounts are there and 3 gift vouchers have to be selected.

This problem involves counting of 3-combinations with repetition from a set of four elements.

Thus we have, n = 4, r = 3

Number of combinations = ${}^{n+r-1}C_r = {}^{4+3-1}C_3$

$$= {}^{6}C_{3} = \frac{6\times 5\times 4}{3\times 2\times 1} = 20$$

Let us Recollect Formulas of Permutation and Combination

S. No.	Type	Repetition	Formula
1.	Permutation of n distinct objects taken r at a time	Not allowed	${}^{n}P_{r} = \frac{n!}{(n-r)!}$
2.	Permutation of n distinct objects taken r at a time	Allowed	n^r
3.	Combination of n distinct objects taken r at a time	Not allowed	${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$
4.	Combination of n distinct objects taken r at a time	Allowed	${}^{n+r-1}C_{r} = \frac{(n+r-1)!}{r!(n-1)!}$

Note: A word of Advice!

When dealing with counting problem, we should think is order important in the problem? When order is required we think in terms of permutations, arrangements and fundamental principle of multiplication. When order is not relevant, combination plays a key role in solving the problem.

Miscellaneous Examples

Example 52: How many words, with or without meaning, each of 2 vowels and 3 consonants can be formed from the letters of the word DAUGHTER?

Solution: In the word DAUGHTER, there are 3 vowels and 5 consonants.

The number of ways of selecting 2 vowels out of 3 = ${}^{3}C_{2}$

The number of ways of selecting 3 consonants out of 5 = ${}^{5}C_{3}$

Therefore, the number of combinations of 2 vowels and 3 consonants is

$${}^{3}C_{2} \times {}^{5}C_{3} = {}^{3}C_{1} \times {}^{5}C_{2} = 3 \times 10 = 30$$

Now each of these combinations has 5 letters which can be arranged among themselves in 5! ways.

Therefore the required number of words is $30 \times 5! = 3600$

Example 53: How many numbers greater than 2000000 can be formed using the digits 1, 3, 0, 3, 2, 3, 2?

Solution: The numbers have to be greater than 2000000, so they can begin either with 2 or 3.

The numbers beginning with 2 = $\frac{6!}{3!} = 4 \times 5 \times 6 = 120$

Total numbers beginning with 3 = $\frac{6!}{2!\,2!} = \frac{6\times5\times4\times3\times2\times1}{(2\times1)\times(2\times1)} = 180$

The required number of numbers = 120 + 180 = 300

Example 54: From a group of 20 students of an environmental club 9 are to be chosen for an educational tour. There are 3 friends among these students who decide that either all of them will join or none of them will join. In how many ways can be students for the educational tour be chosen?

Solution: Let us consider the case when the three students doesn't join. In that case 9 students have to be chosen from the remaining 17 students, i.e.,

$^3C_0 \times {}^{17}C_9 = {}^{17}C_9$ ways

In case the three students join the tour we chose the remaining 6 students from 17 students, which can be done in $^3C_3 \times {}^{17}C_6 = {}^{17}C_6$ ways.

Total number of combination satisfying the given condition = $^{17}C_9 + {}^{17}C_6$

= 24310 + 12376

= 36,686

Example 55: Find the number of (i) combinations and (ii) permutations of the letters of the word ACCOUNTANCY taken 4 at a time.

Solution: The given letters of the word are

(CCC), (AA), (NN), O, U, T and Y. There are 11 letters of which 7 are distinct.

The various choices of 4 letters at a time are

(i) 3 identical and 1 different letter

$^3C_3 \times {}^6C_1 = 1 \times 6 = 6$ ways

(ii) 2 pairs of identical letters

$^3C_2 = 3$ ways (i.e., CCAA, CCNN, AANN)

(iii) One pair of identical letters and 2 different letters.

$^3C_1 \times {}^6C_2 = 3 \times \frac{6 \times 5}{2 \times 1} = 45$ ways

(iv) 4 different letters.

$7C4 = \frac{7 \times 6 \times 5 \times 4}{4 \times 3 \times 2 \times 1} = 35$ ways

So total number of combinations = 6 + 3 + 45 + 35 = 89

To find the permutations we have to arrange the letters in all the above cases:

(i) $6 \times \frac{4!}{3!} = 24$

(ii) $3 \times \frac{4!}{2! \times 2!} = 18$

(iii) $45 \times \frac{4!}{2!} = 540$

(iv) $35 \times 4! = 840$

The total number of permutations are = 24 + 18 + 540 + 840 = 1422

Example 56: There are 8 members in a committee. In how many ways we can choose:

(i) a subcommittee consisting of 3 members?

(ii) a chairperson, a secretary and a treasurer assuming that one person cannot hold more than one position?

Solution:

(i) To form a subcommittee we have to choose 3 members out of 8 which can be done in $^8C_3 = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$ ways

(ii) To choose a chairperson, a secretary and a treasurer the order is also important, so it can be done in $^8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 8 \times 7 \times 6 = 336$ ways

Example 57: A company has 5 senior officers and 7 junior commissioned officers (JCO's). A team of 4 is to be sent on a special mission. In how many ways it can be formed so that it comprises of

(i) any 4 officers?

(ii) 4 senior officers?

(iii) 2 senior and 2 junior officers (JCO's)

(iv) at least 2 senior officers

Solution:

(i) Any 4 officers can be selected in $^{12}C_4 = 495$ ways

(ii) 4 senior officers can be chosen in $^{5}C_4 = 5$ ways.

(iii) 2 senior and 2 junior officers can be selected in check for sub-script

$$^{5}C_2 \times {}^{7}C_2 = \frac{5\times4}{2\times1} \times \frac{7\times6}{2\times1} = 10 \times 21 = 210 \text{ ways}$$

(iv) The team of At least 2 senior officers can constitute of 2S and 2J or 3S, 1J or 4S

$\Rightarrow {}^{5}C_2 \times {}^{7}C_2 + {}^{5}C_3 \times {}^{7}C_1 + {}^{5}C_4$

$= \frac{5\times4}{2\times1}\times\frac{7\times6}{2\times1}+\frac{5\times4}{2\times1} \times 7 + 5$

$= 10 \times 21 + 70 + 5$

$= 210 + 70 + 5 = 285$

Example 58: From a total of 9 players a basketball team of playing 5 is to be selected. How many teams are possible if

(i) the distinct positions of the playing 5 are to be taken into consideration.

(ii) the distinct position of the playing 5 are not taken into consideration.

(iii) the distinct positions are not taken into consideration, but two players either Krish or Rohit (but not both) should be in the playing 5.

Solution:

(i) If the distinct position of the playing 5 are to be taken into consideration, then number of ways = $^{9}P_5 = \frac{9!}{4!} = 15120$

(ii) If the distinct position of the playing 5 are not taken into consideration, then number of ways = ${}^{9}C_5$ = 126

(iii) Number of ways, when either Krish or Rohit (but not both) are in the playing 5

$= 2 \times {}^{1}C_1 \times {}^{7}C_4$

$= 2 \times 1 \times 35 = 70$

Example 59: Quality Control:

A medical store receives a shipment of 24 infrared temperature guns, including 5 that are defective. Three of these guns are to be sent to a private hospital.

(i) How many selections can be made

(ii) How many of these selections will contain no defective guns?

Solution:

(i) Number of selections = ${}^{24}C_3$ = 2024

(ii) Number of selections which contains no defective guns = ${}^{19}C_3$ = 969

Miscellaneous Exercise 1.5

1. An investment banker finalises the list of specific entities worthy of investment in each of the 3 instruments given below:

   3 private limited companies for direct equity investment

   5 mutual fund schemes

   2 banks for making fixed deposits

   In how many ways the investment can be made if:

   (i) the banker decides to invest the entire fund only in entity

   (ii) the banker chooses to invest in one entity of each of the three instruments.

2. A cookie shop has five different kind of cookies. How many different ways can six cookies be chosen assuming that only the type of cookie and not the individual cookies or the order in which they are chosen matters.

3. In an examination, a question paper consists of 12 questions divided into two sections i.e. A and B, containing 7 and 5 questions, respectively. A student is required to attempt 8 questions in all and first question of section A is compulsory. In how many ways can be student select the questions if at least 3 questions are to be attempted from each section.

4. How many 4 digit numbers can be formed from the digits 1, 1, 2, 2, 3, 3, 4 and 5?

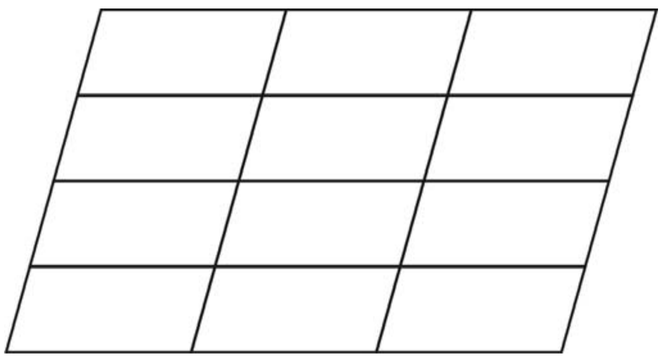
5. How many 5 letter word can be formed using 3 letters of the word ALGORITHM and 2 letters from the world DUES.

6. Life Sciences (Medicine): There are 8 standard classification of blood type. An examination for prospective laboratory technicians consists of having each candidate determine the type of 3 blood samples.

   (i) How many different examinations can be given if no. 2 samples are of the same type.

   (ii) How many different examination papers can be given if 2 or more samples can have the same type.

7. Find the number of parallelograms in the following figure.



8. The number of incorrect predictions of 4 successive football matches is

   (i) 81 (ii) 64

(iii) 80 (iv) 63

9. Number of ways in which 15 different children can sit in a merry-go-round relative to one another is

(i) 1/2 (14!) (ii) 14!

(iii) 1/2 (15!) (iv) $2 \times 14!$

10. Number of diagonals of a convex hexagon are:

(i) 3 (ii) 6

(iii) 9 (iv) 15

11. Number of divisors of 10,000,000 are:

(i) 7 (ii) 8

(iii) 49 (iv) 64

12. A donuts shop offers 20 kinds of donuts. The shop has at least a dozen donuts of each kind if person enters the shop he can select dozen donuts in

(i) ${}^{31}C_{12}$ ways (ii) ${}^{30}C_{12}$ ways

(iii) ${}^{32}C_{12}$ ways (iv) 240 ways

13. The number of permutations of n different things taken r at a time in which m particular things are placed in m given places in definite order is:

(i) ${}^{n-m}P_{r-m} \times m!$ (ii) $(n - m + 1)!$

(iii) ${}^{n-m}P_{r-m}$ (iv) ${}^{n}P_{r} - m!$

Answers:

1. (i) 10 (ii) 30
2. ${}^{5+6-1}C_6 = 210$
3. 265 ways

4. 354

5. ${}^{9}C_3 \times {}^{4}C_2 \times 5! = 60480$

6. (i) 56 (ii) 512

7. ${}^{4}C_2 \times {}^{5}C_2 = 60$

8. (iii)

9. (ii)

10. (iii)

11. (iv)

12. (i)

13. (iii)

Summary

- There are two basic principles of counting

  1) **The Rule of Sum** (Fundamental Principle of Addition)
  If an event E can occur in m ways and another event F can occur in n ways, further the two events cannot occur simultaneously, then the either event E or F can occur in (m+n) ways.

  2) **The Rule of Product** (Fundamental Principle of Multiplication)
  If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is mxn.

- **Factorial**

  n! = 1 x 2 x 3 x … x (n-1) x n

  n! = n x (n-1) !

- The number of permutations of n different objects taken r at a time, where o < r ≤ n and objects do not repeat is given by ${}^{n}P_r = \dfrac{n!}{(n-r)!}$ where o ≤ r ≤ n.

- The number of permutations of n different objects taken r at a time, when repetition is allowed is n^r.

Chapter VII

Mathematical And Logical Reasoning

Learning Objectives

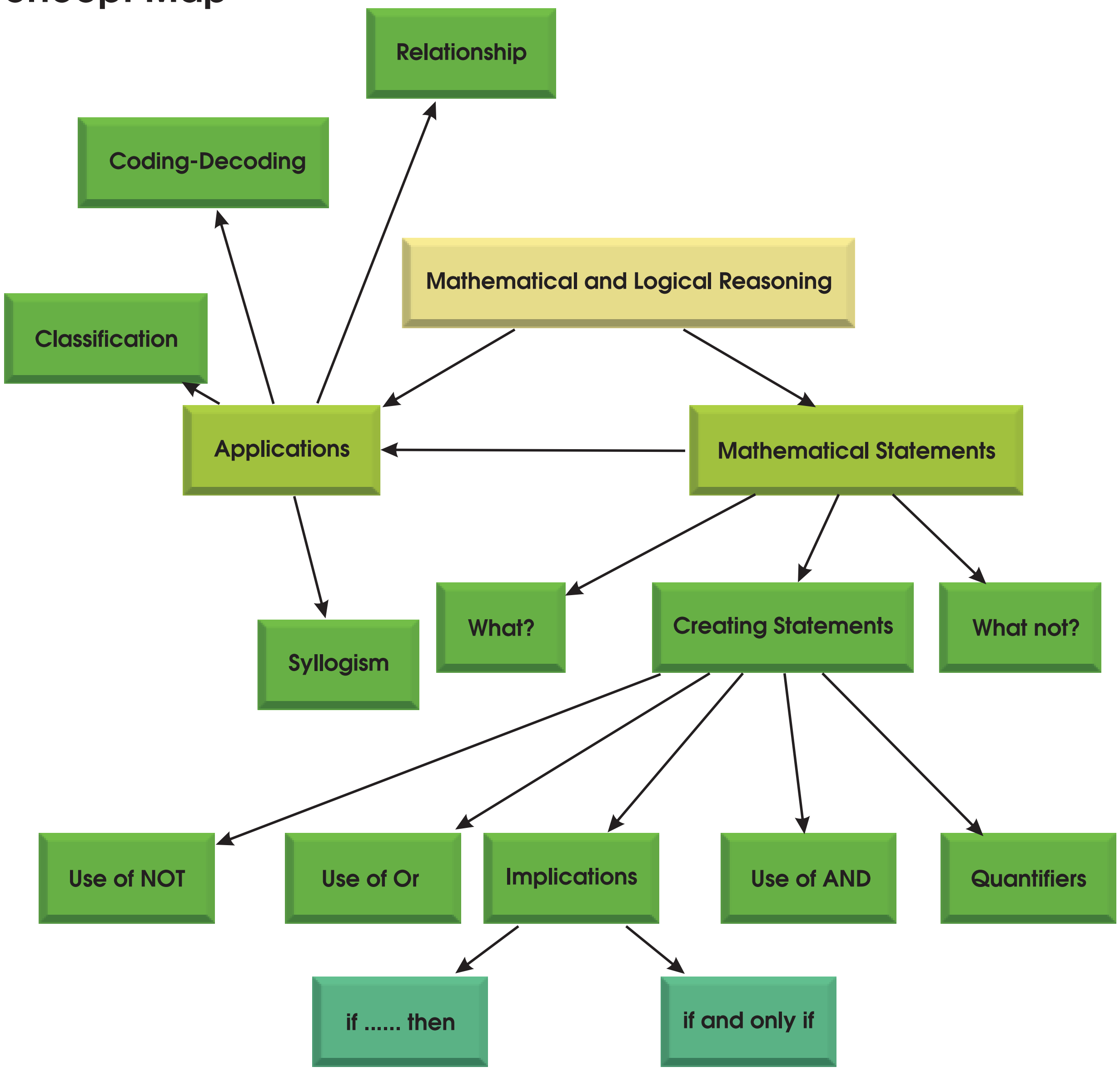
After completion of this unit student will be able to

- Identify mathematically acceptable statements
- express the implications of the compound statements
- validate mathematical statements
- solve logical problems involving syllogism, coding-decoding, odd man out and relationship

Before you start you should know

The fundamentals of algebra and basics of mathematical language studied upto class X.

Concept Map



7.1 Introduction

Deductive reasoning determines whether the truth of a conclusion can be determined, based solely on the truth of the premises. For example: "When it rains, things outside get wet. Thus we can conclude: The grass is outside, therefore: when it rains, the grass gets wet. Mathematical logic and logical reasoning are commonly associated with this type of reasoning. In this unit, we will explore the basics of deductive reasoning such as mathematically acceptable statements, compound statements with 'And' 'Or', 'Not', 'If and only If' and 'If…then'. We will also learn how to check validity of a mathematical statement. Logical functions, Not, If, Or and And are also explored using a spreadsheet as practical work.

Mathematical logic is a subfield of mathematics exploring the applications of formal logic to mathematics. Hence we will also learn different situations to understand reasoning using logic. This unit also gives the ability to understand the statements using syllogism, to find out relationship using blood relationship, to decode and the ability to classify using odd man out.

7.2 Mathematical Statements

Read the following sentences and classify them:

1. He stopped.
2. Did he stop?
3. I say, stop.
4. How he stopped!

You will notice that these are examples of four types of sentences in English language, classified by their purposes. They are:

1. declarative sentence which is a statement
2. interrogative sentence which is a question
3. imperative sentence which is a command
4. exclamative sentence which is an exclamation

Thus we can see that a statement is a declarative sentence. Declarative sentences tell us something or in other words, they give us information. Now read the following sentences and check whether they are true or false.

1. An apple weighs more than a human being.
2. Those who study mathematics are intelligent.
3. Capital of India is New Delhi.

When we read these sentences, we immediately decide that some sentences are false, some are true and in certain cases we are not sure as they are ambiguous. We find that there is no confusion regarding some sentences. In mathematics sentences that result in a true or a false are called mathematically acceptable statements. Thus the sentence "Capital of India is New Delhi" and " An apple weighs more than a human being" are the mathematically acceptable statement or mathematical statements. The sentence "Those who study mathematics are intelligent" is an ambiguous statement as intelligence is a relative term and nothing definite can be stated about intelligence of the students who study mathematics.

A sentence is called a mathematically acceptable statement, or simply a statement, if it is either true or false but not both.

While studying mathematics, we come across many such sentences. Some examples are:

1. Two plus two equals four.
2. The sum of two positive numbers is positive.
3. All prime numbers are odd numbers.

Of these sentences, the first two are true and the third one is false. There is no ambiguity regarding these sentences. Therefore, they are mathematical statements. Now consider the following sentence:

The sum of x and y is greater than 0

Here, we are not in a position to determine whether it is true or false, unless we know what are x and y? For example, it is false when $x = 1$ and $y = -3$ and true when $x = 1$ and $y = 0$. Therefore, this sentence is not a statement. But the sentence: **For any natural numbers x and y, the sum of x and y is greater than 0 is a statement.**

While dealing with statements, we usually denote them by small letters p, q, r,...

For example, we denote the statement "Fire is always hot" by p. This is also written as

p: Fire is always hot.

If you wish, It can also be written as

q: Fire is always hot.

Check your Progress-1

Check whether the following sentences are statements. Give reasons for your answer.

1. 8 is less than 6.
2. Every set is a finite set.
3. The sun is a star.
4. Mathematics is fun.
5. There is no rain without clouds
6. How far is Chennai from here?

7.3 Creating new Statements

We can create new statements from the existing ones by negation or by connecting statements. Let us see how to use negation and make compound statements or split a compound statement to make multiple statements.

a) Using Negation

The denial of a statement is called the negation of the statement.Let us consider the following statement:

p: New Delhi is a city.

The negation of this statementcan be in any of the following forms.

- It is not the case that New Delhi is a city
- It is false that New Delhi is a city.
- New Delhi is not a city.

You can very well see that negation of a statement is also a statement which

If p is a statement, then the negation of p is also a statement and is denoted by ~ p, and read as 'not p'.

Let

r : There does not exist a quadrilateral which has all its sides equal.

Then negation of this statement will be:

~ r : There exists a quadrilateral which has all its sides equal.

The new statement is true because we know that square is a quadrilateral such that its four sides are equal. Note that negation of a statement can either be true or false. If a statement is true, its negation will be false and if a statement is false, its negation will be true. We can show this fact in a tabular form as given below.

Negation:

Statement A	**Statement B as negation of statement A**
True	False
False	True

Compound Statements:

Many mathematical statements are obtained by combining one or more statements using some connecting words like "**and**", "**or**", etc.

Compound Statement is a statement which is made up of two or more statements. In this case, each statement is called a component statement.

Let us take an example:

The sky is blue and the grass is green.

This is a compound statement and is made up of two statements p and q.

p: The sky is blue.

q: The grass is green.

p and q are called the component statements.You can also combine these two component statements and make a compound statement using another connecting word, say **or**. Now the compound statement would be:

The sky is blue or the grass is green.

The connecting words which are often found in compound statements are "**and**","**or**", "**if.... then ...**" "**if and only if**". These are called **connectors.** When we use these compound statements, it is necessary to understand the role of these words

(i) **Compound Statements with 'and'**

Let us take a compound statement: **p: A point occupies a position and its location can be determined.** The statement can be broken into two component statements as :

q: A point occupies a position.

r: Its location can be determined.

Here, we observe that both statements are true. Let us look at another compound statement.

a: 42 is divisible by 5, 6 and 7. This statement has following component statements.

b: 42 is divisible by 5.

c: 42 is divisible by 6.

d: 42 is divisible by 7.

Here, we know that the compound statement a is false, while component statements **c and d** are true. We can conclude the following:

- The compound statement with 'and' is true, if all its component statements are true.
- The compound statement with 'and' is false if any of its component statement is false.

Let us now see whether the above conclusion is true in the following cases.

A line is straight and extends indefinitely in both the directions. The component statements of this compound statement are

p: A line is straight.

q: A line extends indefinitely in both the directions.

Both these statements are true, therefore, the compound statement is true.

For a compound statement: **0 is less than every positive integer and every negative integer;** the component statements are

m: 0 is less than every positive integer.

n: 0 is less than every negative integer.

The second statement is false. Therefore, the compound statement is false.

If the statement is: **All living things have two legs and two eyes;** its two component statements are:

p: All living things have two legs.

q: All living things have two eyes.

Both these statements are false. Therefore, the compound statement is also false. We can represent these facts in a tabular form as under:

Statement A	Statement B	Compound statement with 'and'
True	True	True
True	False	False
False	True	False
False	False	False

Remember: A statement with a word "and" cannot be always considered as a compound statement

Let us see the following statement.

m: A mixture of alcohol and water can be separated by chemical methods.

This sentence cannot be considered as a compound statement with "and". Here the word "and" refers to two things - alcohol and water and is not used for connecting two statements.

(ii) Compound Statements with 'or'

Let us consider the following statement:

p: Two lines in a plane either intersect at one point or they are parallel.

This means that if two lines in a plane intersect, then they are not parallel. Alternatively, if the two lines are not parallel, then they intersect at a point. i.e, this statement is true in both the situations. In order to understand statements with "Or" we first notice that the word "Or" is used in two ways in English language. Let us first look at the following statement.

p: An ice cream or 'lassi' is available with a 'thali' in a restaurant.

This means that a person who does not want ice cream can have a 'lassi' along with a 'thali' or one who does not want 'lassi' can have an ice cream along with a 'thali'. A person cannot have both ice cream and lassi. This is called an exclusive "**Or**". Here is another statement.

A student who has taken Biology or Chemistry in graduation can apply for M.Sc. Microbiology programme.

Here we mean that the students who have taken both biology and chemistry can apply for the Microbiology programme, as well as the students who have taken only one of these subjects can apply for the Microbiology programme. In this case, we are using **inclusive** "**Or**". If we tabulate the outcomes of combining two statements using 'or' in inclusive or exclusive ways, we will get the following:

Statement A	Statement B	Compound statement with 'or' which is exclusive
True	True	False
True	False	True
False	True	True
False	False	False

For **exclusive or** we can state:

1. A compound statement with an 'Or' is true when one of the component statements is true
2. A compound statement with an 'Or' is false when both the component statements are true or both statements are false.

Inclusive or

Statement A	Statement B	Compound statement with 'or' which is inclusive
True	True	True
True	False	True
False	True	True
False	False	False

For **inclusive or** we can state:

1. A compound statement with an 'Or' is true when one component statement is true or both the component statements are true.
2. A compound statement with an 'Or' is false when both the component statements are false.

Check your Progress-2

For each of the following statements, determine whether an inclusive "Or" or exclusive "Or" is used. Give reasons for your answer.

1. The school is closed if it is a holiday or a Sunday.

2. Two lines intersect at a point or are parallel.

3. Students can take French or Sanskrit as their third language.

7.4 Quantifiers

Quantifiers are phrases used in statements like, "**there exists**" and "**for all**". For example, consider the statement.

p: There exists a rectangle whose all sides are equal.

This means that there is at least one rectangle whose all sides are equal. A word closely connected with "there exists" is "for every" (or for all). For example:

p: For every prime number p, $\sqrt{p}$ is an irrational number.

In general, a mathematical statement that says "for every" can be interpreted as saying that all the members of the given set S where the property applies must satisfy that property.

7.5 Implications-'If-then'

An implication-'If-then' is the compound statement of the form "if p then q" Let us consider the statement: **If you are born in some country, then you are a citizen of that country.** When we look at this statement, we observe that it corresponds to two statements p and q given by

p : You are born in some country. q : You are citizen of that country.

Then the sentence "**if p then q**" says that in the event if p is true, then q must be true.

p implies q is denoted by $p \Rightarrow q$. The symbol $\Rightarrow$ stands for implies.

Important aspects of If-then statement:

1. "if p then q" is that, **it does not say anything on q when p is false.** In the given example, if you are not born in the country, then you cannot say anything about q.

2. "if p then q" is that, the statement **does not imply that p happens.** That means if you are a citizen of a country does not imply that you are born in that country.

Contrapositive Statement : A contrapositive statement can be formed from a given statement with "**if-then**". For example, let us consider the following "if-then" statement.

If the physical environment changes, then the biological environment changes.

The contrapositive of this statement is

If the biological environment does not change, then the physical environment does not change.

More examples:

1. If a number is divisible by 9, then it is divisible by 3.
2. If you are born in India, then you are a citizen of India.
3. If a triangle is equilateral, it is isosceles.

The contrapositive of the these statements are

1. If a number is not divisible by 3, it is not divisible by 9.
2. If you are not a citizen of India, then you were not born in India.
3. If a triangle is not isosceles, then it is not equilateral.

Contrapositive of the statement : if p then q is "if ~ q then ~ p"

Converse Statement: A **converse** statement can be formed from a given statement with "if-then". The converse of a given statement "**if p then q**" **is "if q then p".**

Let us take an example. For the statement: **p: If a number is divisible by 10, it is divisible by 5.**

the converse is **q: If a number is divisible by 5, then it is divisible by 10.**

7.6 Implications 'If and only if'

Given below are two pairs of statements.

p: If a rectangle is a square then all its four sides are equal.

q: If all the four sides of a rectangle are equal then the rectangle is a square.

If we combine these two statements using "if and only if " we will get:

r: A rectangle is a square if and only if all its four sides are equal.

'If and only if', represented by the symbol '$\Leftrightarrow$' means the following equivalent forms for the given statements p and q.

- p if and only if q
- q if and only if p
- p is necessary and sufficient condition for q and vice-versa
- p $\Leftrightarrow$ q

7.7 Validating Statements

Validating a statement is to check whether a statement is true. Validation of a statement depends upon which of the special words and phrases "and", "or", and which of the implications "if and only if", "if-then", and which of the quantifiers "for every", "there exists", appear in the given statement.

Let us summarize what we have learnt and state as rules for validation. This will be useful in our next section on application of logical reasoning.

Rule 1

Compound statement with 'and'

If p and q are mathematical statements, then in order to show that the statement "p and q" is true, the following steps are followed.

Step-1 Show that the statement p is true.

Step-2 Show that the statement q is true.

Rule 2

Compound Statements with "or" (exclusive or)

If p and q are mathematical statements, then in order to show that the statement "p or q" is true, one must consider the following:

Case 1 By assuming that p is false, show that q must be true.

Case 2 By assuming that q is false, show that p must be true.

Rule 3

Implications "If-then"

Case 1 By assuming that p is true, prove that q must be true.(Direct method)

Case 2 By assuming that q is false, prove that p must be false.(Contrapositive method)

Rule 4

Implications "if and only if "

In order to prove the statement "p if and only if q", we need to show.

Step-1 If p is true, then q is true.

and

Step-2 If q is true, then p is true.

7.8 Applications of Reasoning

Let us now use the learning about the statements and connectives in developing our reasoning ability. It is expected that you are able to solve questions of reasoning of categorical syllogism, coding-decoding, blood relationship and odd man out. This will help us in development of logical reasoning by enhancing our ability to classify, find patterns and relationship, analyse etc. Theoretical aspects given below are meant just to set the context.

Note to teachers: The assessment should be restricted to problems based on categorical syllogism, coding-decoding, blood relationship and odd-one out. The classroom transactions should focus on the development of logical thinking based on these examples without much focus on the technical terms given in the unit or rote learning of concepts. The explanation is given only for setting the context.

a) Categorical Syllogism

We have learned about mathematical statements and the rules for validation in the earlier part of this unit. Let us make use of this learning in understanding syllogism. Syllogism is a word derived from a Greek word syllogismos meaning "conclusion or inference". It is a kind of logical argument that applies deductive reasoning to arrive at a conclusion based on two or more propositions that are asserted or assumed to be true. Categorical syllogisms consist of three categorical propositions or parts as given below:

- Major premise
- Minor premise
- Conclusion

Each of the premises has one term in common with the conclusion. In a major premise, this is the major term (i.e., the predicate of the conclusion). In a minor premise, this is the minor term (i.e., the subject of the conclusion).

For example:

Major premise: All humans are mortal.

Minor premise: All Indians are humans.

Conclusion: All Indians are mortal.

Each of the three distinct terms represents a category. In the above example, humans, mortal, and Indians are three different categories. Mortal is the major term and Indians is the minor term. The premises also have one term in common with each other, which is known as the middle term. In this example, humans is the middle term.

Major premise: All mortals die.

Minor premise: All men are mortal.

Conclusion: All men die.

Here, the major term is die, the minor term is men, and the middle term is mortals.

Let us explore syllogism with some examples.

Two statements are given below followed by two conclusions numbered as I and II respectively. Consider the given statements as true even if they seem to be not. After reading all the conclusions, confirm which of the given conclusions logically follows, disregarding commonly known facts.

Q 1 - Statements:

I. Some goats are vegetarians.

II. All vegetarians are blessed.

Conclusions:

I. Some goats are blessed.

II. At least some blessed are vegetarians.

A - only conclusion I follows.

B - only conclusion II follows.

C - either conclusion I or II follows.

D - neither conclusion I nor II follows.

E - both conclusion I and II follow.

Answer : E

Explanation : It is given that some goats are vegetarians and all vegetarians are blessed. So we can say that some goats are blessed. Hence conclusion I follows. Also all vegetarians are blessed. Thus some blessed are vegetarians. Hence conclusion II follows.

Q 2 - Statements:

I. Some pictures are beds.

II. All beds are trees.

Conclusions:

I. Some pictures are trees.

II. At least some trees are beds.

A - only conclusion I follows.

B - only conclusion II follows.

C - either conclusion I or II follows.

D - neither conclusion I nor II follows.

E - both conclusion I and II follow.

Answer : E

Explanation : It is given that some pictures are beds and all beds are trees. Thus we can say that some pictures are trees. Hence conclusion I follows. Also all beds are trees. Thus some trees are beds. Hence conclusion II also follows.

Check your Progress-3

In each of the following questions two statements are given and these statements are followed by two conclusions numbered (1) and (2).

You have to take the given two statements to be true even if they seem to be at variance from commonly known facts. Read the conclusions and then decide which of the given conclusions logically follows from the two given statements, disregarding commonly known facts.

Give answer:

- (A) If only conclusion (1) follows.
- (B) If only conclusion (2) follows.
- (C) If either (1) or (2) follows.
- (D) If neither (1) nor (2) follows.
- (E) If both (1) and (2) follow.

1. **Statements:** Some actors are singers. All the singers are dancers.

Conclusions:

1. Some actors are dancers.
2. No singer is actor.

2. **Statements:** All the harmoniums are instruments. All the instruments are flutes.

Conclusions:

1. All the flutes are instruments.
2. All the harmoniums are flutes.

3. **Statements:** Some mangoes are yellow. Some fruits are mangoes.

Conclusions:

1. Some mangoes are green.
2. Fruits are yellow.

4. **Statements:** Some ants are parrots. All the parrots are apples.

Conclusions:

1, All the apples are parrots.

2. Some ants are apples.

b) Coding - Decoding

Coding is a process used to encrypt a word, a number in a particular code or pattern based on some set of rules. Decoding is a process used to decrypt the pattern into its original form from the given codes. This process has tremendous application in data management, security, computing etc.

Coding-Decoding tests the student's ability to find out the rule under which a message has been coded and to decipher the message in the original form.

There are many types of challenges in coding and decoding. We will discuss few types here.

Type 1: letters

Example1 : If DPT is EQU then what is TRY?

Explanation: If we observe carefully then we notice E is the letter next to D , Q next to P and U next to T, applying the same logic for TRY, we get the answer as USZ.

Ans1. USZ

Example 2: In a certain language, if BLOWN is coded as BLNOW then how will RIGHT be coded?

a) HIRGT

b) SJHIU

c) GHIRT

d) THIGR

Explanation: The letters of the word BLOWN are arranged in ascending order of English alphabets, similarly if we arrange the letters of the word RIGHT the arrangement will be GHIRT.

Ans2. c) GHIRT

Type 2: Numbers

Let us assign a number to each alphabet. Then each letter can be equated to a number as given below.

1	2	3	4	5	6	7	8	9	10	11	12	13
A	B	C	D	E	F	G	H	I	J	K	L	M
14	15	16	17	18	19	20	21	22	23	24	25	26
N	O	P	Q	R	S	T	U	V	W	X	Y	Z

Example 3: In a certain language, PAC is coded as 61, how will NEP be coded?

a) 40

b) 66

c) 80

d) 46

Explanation: If we arrange the English alphabets in reverse order then the positions of P, A and C are 11, 26 and 24 respectively. When we add these numbers we get 61. Similarly, when we add the reverse position numbers of the letters of the word PEN we get, 13+22+11 i.e. 46.

Ans 3. d) 46

Type 3: Symbols

Example 4: If MADAM is coded as ✻?#?✻ and DOOM is coded as #%%✻ then LAD will be coded as ....

a) ?#%

b) &?#

c) ✻#/

d) ??#

Explanation: In this example we observe that letter M is represented by✻ , A by ? , D by # and O by % so LAD will be coded as &?#.

Ans 4. b) &?#

Example 5: If BRED is coded as #✻?%, then BREEZE will be coded as ....?

a) #&%%?✻

b) ??%%#✻

c) %%%?^^

d) ???%#@

Explanation: In this example we observe that the letters of the word BRED can be represented by any of the four symbols #,⁎,? or %. When we observe the letters of the word BREEZE we find that E is repeating three times so the same symbol should appear thrice in the answer So we can zero down our answer to option c) or d) but no other letter is getting repeated in the word so all other symbols should be different. Hence our answer is ???%#@.

Ans 5. d) ???%#@

Check your Progress-4

1. **In a certain language, if CAMEL is coded as DBNFM then how will ROOM be coded?**

   a) NPPQ

   b) SPPN

   c) PPON

   d) NQQP

2. **If BLEPIN is coded as 987416, MATPIN is coded as 123416, then TABLE is coded as?**

   a) 32987

   b) 32897

   c) 38987

   d) 21987

3. **In a certain language,**

   'hi mi si' means 'Air is life'.

   'si ni zi ' means 'Balloon of Air'.

   'ci zi mi' means 'Life of PI'.

   Which of the following represents 'PI' in that language?

a) hi

b) mi

c) ci

d) si

4. In a certain language,

321 means 'Glass of Tea'.

426 means 'Tea is Brown'.

796 means 'Trunks are Brown'.

Which of the following represents 'is' in that language?

a) 6

b) 7

c) 4

d) 2

c) Blood Relation:

Finding out relationship between variables is very important in analyzing different situations. We can enhance this ability using the concept of Blood relations. Blood relation shows the different relations among the members of a family. Based on the information given, you are required to find the relation between particular members of the family.

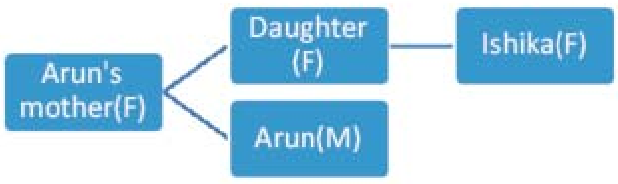
To solve the problems based on blood relation it is always better to draw the diagram or family tree on the basis of information given in the question. Each generation should be drawn at different levels of the diagram. These problems are based on blood relations strictly, so they do not include information about friends etc. Let's take a look at some important blood relations:

Father of father or mother	Grand father
Mother of father or mother	Grand mother
Brother of Father	Uncle
Sister of father	Aunt
Father of Wife/Husband	Father - in - law
Mother of Wife/Husband	Mother - in - law
Son / Daughter of Uncle/ Aunt	Cousin
Brother/Sister of Husband/Wife	Brother - in - law/Sister - in - law
Husband of Sister	Brother - in - law
Wife of Brother	Sister - in - law
Son / Daughter of Brother / Sister	Nephew/Niece
Brother/Sister of Mother	Maternal Uncle / Maternal Aunt

Please note that above table represents the relationship in English language. In other languages like Hindi there may be different words used for relationship. e.g in Hindi use of term uncle may not be similar to the usage in English.

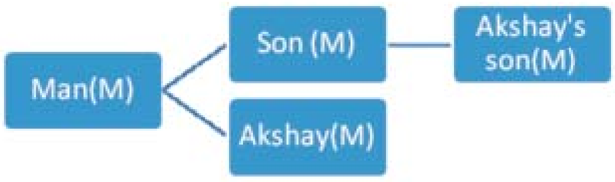
Example 1: Arun said, "Ishika is the daughter of my mother's daughter". How is Arun related to Ishika?

Explanation : Arun's mother's daughter is Arun's sister and her daughter Ishika is Arun's niece. Therefore, Arun is uncle of Ishika. If M stands for male and F stands for female, we can draw :-



Example 2: Pointing to a man, Akshay said," His son is my son's uncle". How is the man related to Akshay?

Explanation: Akshay's son's uncle would be none other but Akshay's brother and his brother's father would be his father too. So, the man is Akshay's father.



Check your progress-5

1. Pointing to a picture the man said, "The lady in the picture is my nephew's maternal grandmother." How is the lady in the picture related to the man's sister who has no other sister?

   a) Cousin

   b) Sister - in - law

   c) Mother

   d) Mother - in law

2. Meenakshi is Kirti's sister. Kaavya is Kirti's mother. Dipesh is Kaavya's father. Esha is Dipesh's mother. Then, how is Meenakshi related to Dipesh?

   a) Grandfather

   b) Grandmother

   c) Daughter

   d) Granddaughter

3. P+Q means P is the brother of Q, R-S means R is the father of S, S/T means S is the sister of T, T x U means T is the mother of U. which of the following means that O is the mother of N?

   a) L+ M/N - O

   b) L - M x O/P

   c) N/M x L /O

   d) M + L / O x N

d) **Odd man out:**

For analysing a situation, one must be able to classify concepts or objects. In the classification one has to identify the common characteristics and distinguishing characteristics. Thus if we are able to enhance our skills in finding out the odd one out, we in turn will be able to enhance our analytical ability. In odd man out problems all the items given in the question except one follow a certain pattern or form a group. The item which is different and doesn't belong to that group will be the answer. Problems of this type are categorized under the head of 'classification'. That means out of all the given elements, one will not fall into the group due to some difference in the property. That is the odd element.

For Example -

Type 1 -

Which one of the words given below is different?

a - Mango

b - Apple

c - Potato

d - Cherry

e - Blackberry

Explanation - Except potato, all the rest are the names of the fruits, while Potato is a vegetable. Hence, it is the odd man. So the answer will be (c)

Type 2 -

Find out the pair which is different.

a - Cow and Buffalo

b - Cock and Hen

c - Horse and Mare

d - Dog and Bitch

e - Peacock and Peahen

From the above, the second one is the feminine of the first one except the first pair. So the answer will be (a).

Check your progress-6

1. Find the odd man out from the given alternatives.
   - a. Insurance
   - b. Provident Fund
   - c. Salary
   - d. Shares
2. Which number is odd man in the numbers 3, 5, 11, 14, 17, 21
   - a. 21
   - b. 17
   - c. 14
   - d. 3
3. Find out the odd man from the figures given below.

a  b  c  d 

7.9 Practical work using spread sheet:

Practical-1: Demonstration of Not or negation

Use the NOT function, one of the logical functions, when you want to make sure one value is not equal to another.

Example

	A	B	C
1	Formula	Description	Result
2	=NOT(FALSE)	Reverses FALSE	TRUE
3	=NOT(TRUE)	Reverses TRUE	FALSE
4	=NOT(1+1=2)	Reverses TRUE	FALSE
5	=NOT(2+2=5)	Reverses FALSE	TRUE

Practical-2: Demonstration of 'and'

The AND Function returns TRUE if all conditions are true and returns FALSE if any of the conditions is false.

Example: Take a look at the AND function in cell D2 below.

D2 =AND(B2>=60,C2>=90)

	A	B	C	D	E	F	G	H	I
1	Name	Score 1	Score 2	Result					
2	Raghu	93	80	FALSE					
3	Jeet	60	91	TRUE					
4	James	58	75	FALSE					
5	Lina	79	94	TRUE					
6	Shalini	41	33	FALSE					
7									

Explanation: The AND function returns TRUE if the first score is greater than or equal to 60 and the second score is greater than or equal to 90, else it returns FALSE.

Practical-3: Demonstration of 'or'

The OR function returns TRUE if any of the conditions is TRUE and returns FALSE if all conditions are false.

Example : Take a look at the OR function in cell D2 below.

D2   fx   =OR(B2>=60,C2>=60)

	A	B	C	D	E	F	G	H	I
1	Name	Score 1	Score 2	Result					
2	Raghu	93	80	TRUE					
3	Jeet	60	91	TRUE					
4	James	58	75	TRUE					
5	Lina	79	94	TRUE					
6	Shalini	41	33	FALSE					
7									

Explanation: The OR function returns TRUE if at least one score is greater than or equal to 60, else it returns FALSE.

Practical-4 : Demonstration of 'If'

The IF function checks whether a condition is met, and returns one value if true and another value if false.

Example: Take a look at the IF function in cell C2 below.

C2   fx   =IF(B2>=60,"Pass","Fail")

	A	B	C	D	E	F	G	H	I
1	Name	Score	Result						
2	Raghu	93	Pass						
3	Jeet	60	Pass						
4	James	58	Fail						
5	Lina	79	Pass						
6	Shalini	41	Fail						
7									

Explanation: If the score is greater than or equal to 60, then IF function returns Pass, else it returns Fail.

Summary

- A mathematically acceptable statement is a sentence which is either true or false.
- Negation of a statement p: If p denotes a statement, then the negation of p is denoted by $\sim$p.
- Compound statements and their related component statements:

  A statement is a compound statement if it is made up of two or more smaller statements. The smaller statements are called component statements of the compound statement.
- A sentence with 'if p then q' can be written in the following ways.
  - p implies q (denoted by $p \Rightarrow q$)
  - p is a sufficient condition for q
  - q is a necessary condition for p
  - p only if q
  - $\sim$q implies $\sim$p
  - The contrapositive of a statement $p \Rightarrow q$ is the statement $\sim q \Rightarrow \sim p$.
- The converse of a statement $p \Rightarrow q$ is the statement $q \Rightarrow p$.
- $p \Rightarrow q$ together with its converse, gives p if and only if q.
- The following methods are used to check the validity of statements:
  - direct method
  - contra positive method
  - method of contradiction
  - using a counter example
- We can use logical reasoning in understanding statements, coding and decoding, in finding out relationship and classifying by finding odd man out.

Reference:

1. Mathematics textbook for class XI-NCERT

Solutions to Check your Progress

Check your Progress-1

Solutions: (Here statement means mathematically acceptable statement)

1. This sentence is false because 8 is greater than 6. Hence it is a statement.
2. This sentence is also false since there are sets which are not finite. Hence it is statement.
3. It is a scientifically established fact that sun is a star and, therefore, this sentence is always true. Hence it is a statement.
4. This sentence is subjective in the sense that for those who like mathematics, it may be fun but for others it may not be. This means that this sentence is not always true. Hence it is not a statement.
5. The sentence is a statement because it results in a true answer.
6. The sentence does not result in a true or a false answer so it is not a statement.

Check your Progress-2

Solutions:

7. Here "Or" is inclusive since school is closed on holiday as well as on Sunday.
8. Here "Or" is exclusive because it is not possible for two lines to intersect and be parallel together.
9. Here also "Or" is exclusive because a student cannot take both French and Sanskrit.

Check your progress-3

Solutions

1. A
2. B

3. D

4. B

Check your progress-4

Solutions

1. B

2. A

3. C

4. C

Check your progress-5

Solutions

1. C

2. D

3. D

Check your progress-6

Solutions

1. C

2. C

3. C

Chapter-VIII

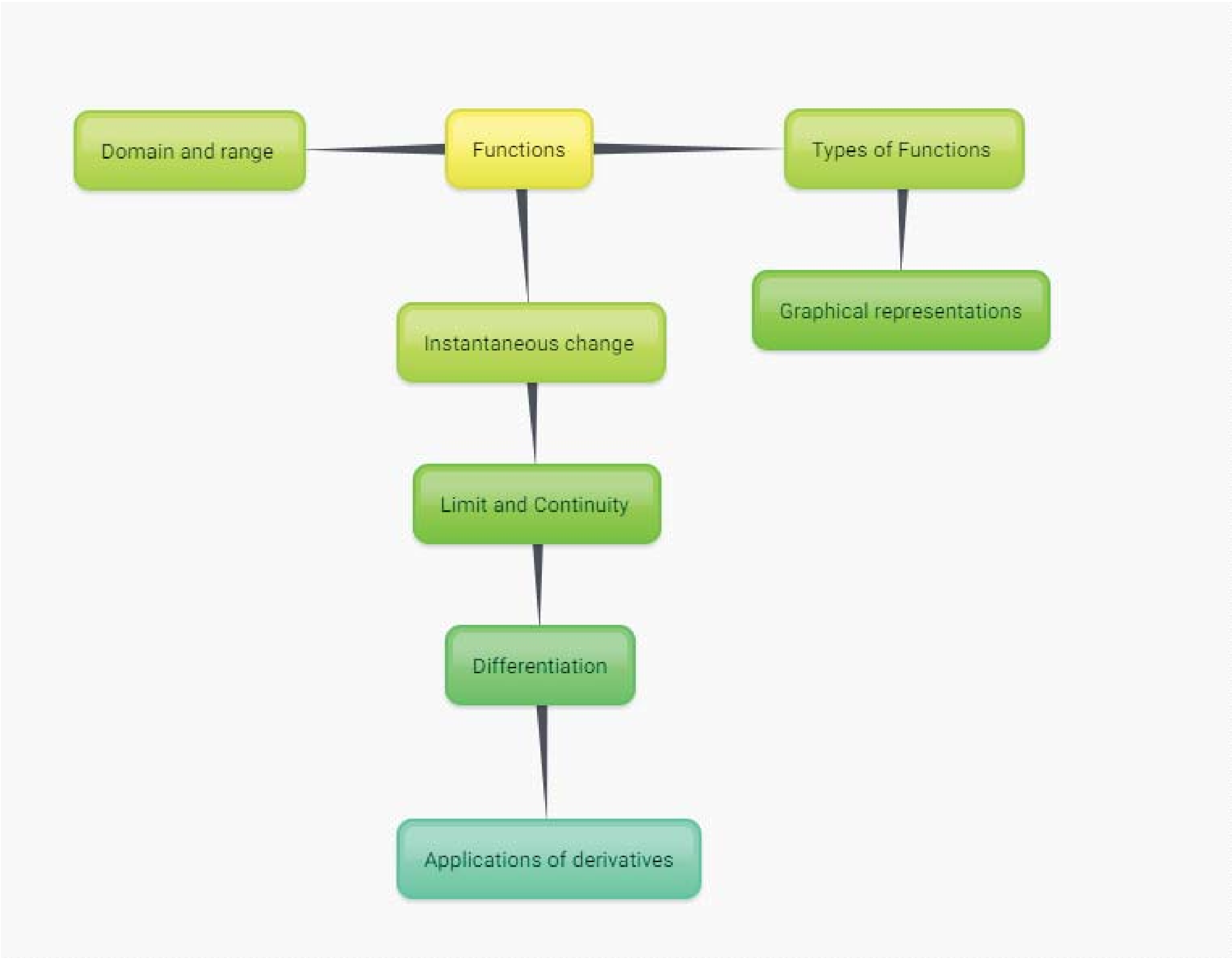
Calculus

Learning Objectives:

After completion of this unit, student will be able to :

- Identify a function and find out domain and range of a function
- Classify functions into different types like Polynomial function, Composite function, Logarithmic function, Exponential function, Modulus function, Greatest integer function, Signum function
- Represent a function graphically and identify the function corresponding to a graph
- Explain the concepts of limit and continuity
- Correlate instantaneous change with differentiation
- Find out derivatives of algebraic functions using first principle, standard formulae and Chain Rule
- find out equations of tangents to the curves, using derivatives

Concept Map



Before you start:

You should know about numbers, polynomials, algebraic expressions and plotting graphs.

8.1 Introduction

Calculus helps us in predictions in the instance of change. For example velocity denotes a change in position with respect to time. We can study this change using Calculus. It provides a way to construct simple quantitative models of change, and to deduce their consequences. By studying Calculus, we will come to know how to control the system to do make it do what we want it to do. The development of Calculus and its applications to Physics and Engineering is probably the most significant factor in the development of modern Science.

Study of changes tend to be simpler if we study it by considering changes over tiny interval of time (instantaneous) rather than studying changes over a period of time.

When we talk about change, we talk about variation of one quantity with respect to another. This can be represented by a function. In this unit, we are going to learn about functions, its types and graphical representations. While studying the instantaneous change, we will develop the idea of instantaneous changing, the concept of limits and continuity of a function. The basic process of differentiation is dealt thereafter with derivatives of algebraic functions and Chain Rule. Tangent lines and equations of tangent are also studied using the concept of differentiation.

8.2 Functions

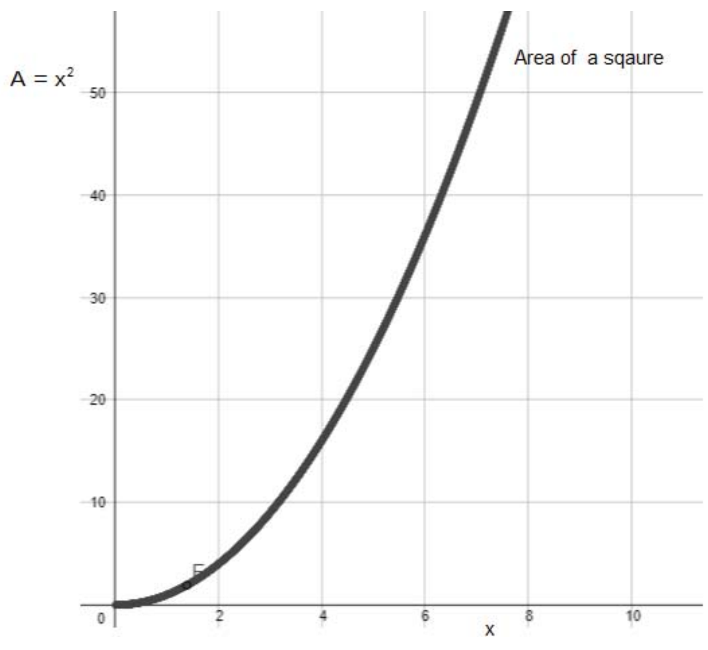
Quantities that change are called **variables.** For example, temperature at a place is a variable as it changes over a period of time. In case of a moving vehicle, speed is a variable. Calculus can be applied to any quantity that varies. In nature if we see that one quantity is connected to another. It means that variation in one quantity affect another. If two variables are connected to each other, one can be seen as a function of another.

Function, in mathematics is an expression, rule, or law that defines a relationship between one variable (the independent variable) and another variable (the dependent variable).

Let us look into the area A of a square. It depends on the length of the side a of square(x). We write $A = x^2$, Here, area (A) is called a dependent variable and it is a function of side of the square (x), which is an independent variable.

There are many ways to express the same function. A function can be represented by an equation, a graph or a table. The function `area of a square' can be expressed with a table. This can also be plotted on a graph with area on the y-axis and side on the x-axis as shown below:

x (cm)	$A = x^2$ (cm^2)
0	0
1	1
2	4
3	9
4	16
5	25



Let us see how to express a function. In the above graph we have seen that A depends on x. Let us write A = f(x) means A is a function of (x)

What will happen if the value of x is negative? Can we have a negative value of length? So the equation of this function will be complete only with some additional information $x \geq 0$.

Instead of using A let us take y as the "dependent variable" and x as the "independent variable". The equation can be written as: $y=x^2$, for $x \geq 0$.

The value of independent variable can be decided by us. The variable y depends on the value of x that we take. This is only a convention. We can also take x as independent and y as dependent variable. In this case, we say that y is a function of x. Hence we can write: y= f(x), for $x \geq 0$.

It can also be written as: $f(x) = x^2$, for $x \geq 0$.

We read that "y equals to x^2 for x greater than or equal to zero. In general, $f(x)=x^2$ is defined for any real number x.

Remember that in a **function**, the **input** value must **have one** and only **one** value for the **output**. Vertical line test is used to determine whether a curve represents a function or not. If any curve cuts a vertical line at more than one points then the curve does not represent a function.

Let us see another function $f(x) = x^2+1$

The output value or value of f(x) when x=0 can be written as $f(0) = 0^2 + 1 = 0 + 1 = 1$.

Similarly we can find out the value of function when the input is x=-1 as below:

$f(-1) = (-1)^2 + 1 = 1+1 = 2$

If you know how to do arithmetic operations of the polynomials you can very well do the same with functions.

Addition	$f(x) + g(x)$	$(f + g)(x)$
Subtraction	$f(x) - g(x)$	$(f - g)(x)$
Multiplication	$f(x) \bullet g(x)$	$(f \bullet g)(x)$
Division	$\frac{f(x)}{g(x)}$ If $g(x) \neq 0$	$\frac{f}{g}(x)$

Illustration:

If $f(x) = 2x+3$ and $g(x) = x^2+3$

Then $f(x)+g(x) = (f+g)(x) = x^2+2x+6$

Where (f+g) is another function which operates on x to get x^2+2x+6

Now try to find out the function (f.g) which product of these two functions.

Check your Progress 1 :

a) State whether y is a function of x in the following two cases. Justify your answer.

(i)

x	*y*
–3	–6
–2	–1
1	0
1	5
2	0

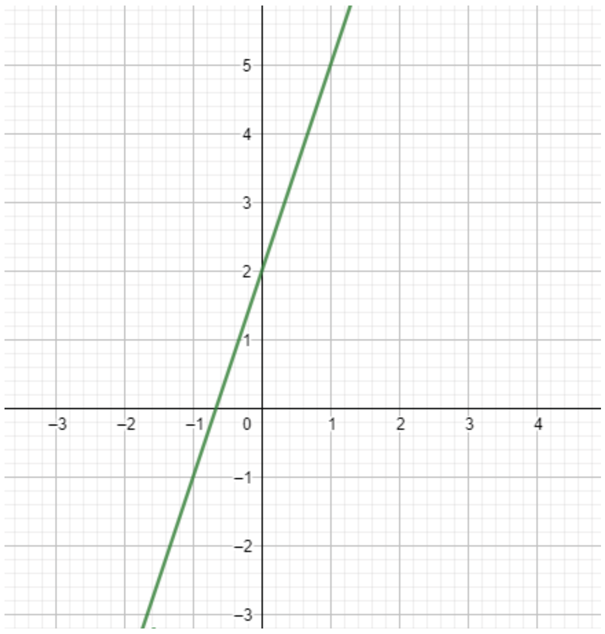
(ii)

x	*y*
–3	4
–2	4
–1	4
2	4
3	4

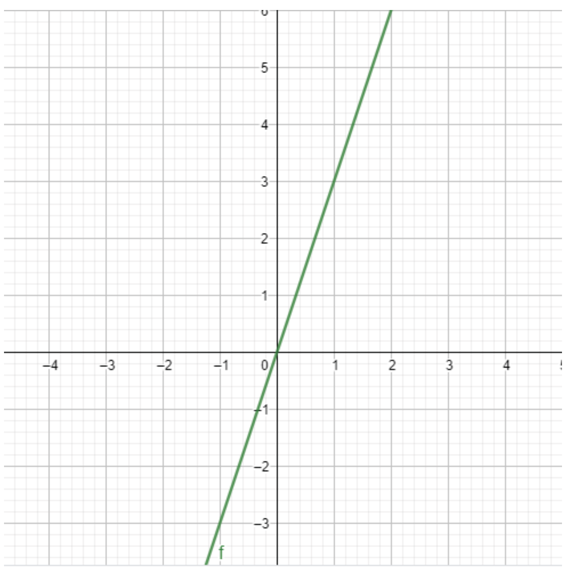
b) If $f(x) = x+1$ and $g(x) = x^2-2x+5$ Find out $(f+g)(x)$. Also plot graphs of $f(x)$, $g(x)$ and $(f+g)(x)$

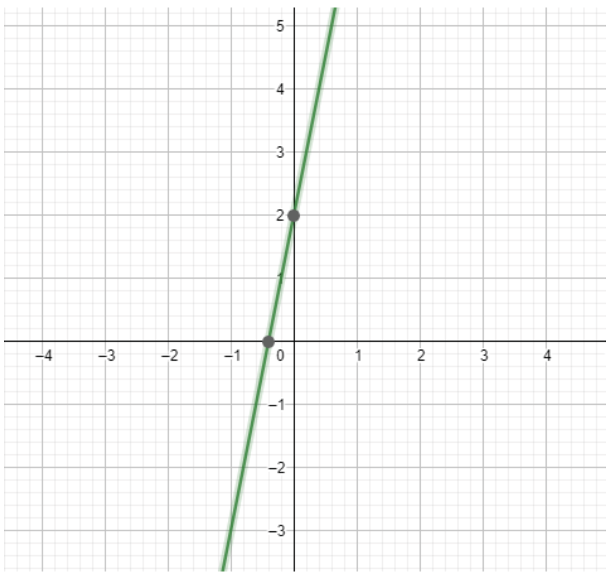
8.3 Graphical representation of functions:

Graphical representation of the function finds many applications. In Finance, Science and Technology graphs are the tools used for many purposes. In the simplest case one variable is plotted as a function of another, typically using rectangular axes. It is therefore important that we understand the nature of graph representing different functions. Let us plot the graph of function f(x) = 3x+2



Graph of f(x) = 3x will be different in its y intercept and if we plot y = 5x+2 will be a straight line with a greater slope.

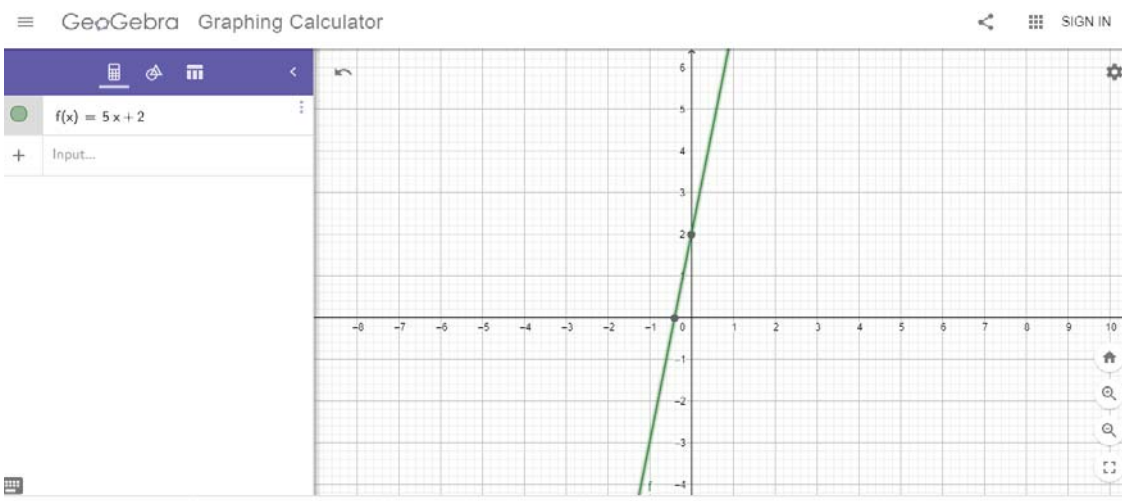




Check your Progress 2 :

Above graphs are plotted using GeoGebra Graphing calculator which is available at : https://www.geogebra.org/graphing?lang=en.

Explore this useful tool in plotting different functions.



Plot the graph of following functions using GeoGebra Graphing calculator:

a) $f(x) = x^2$

b) $f(x) = x^3$

c) $f(x) = 1/x$

8.4 Domain and range of a function:

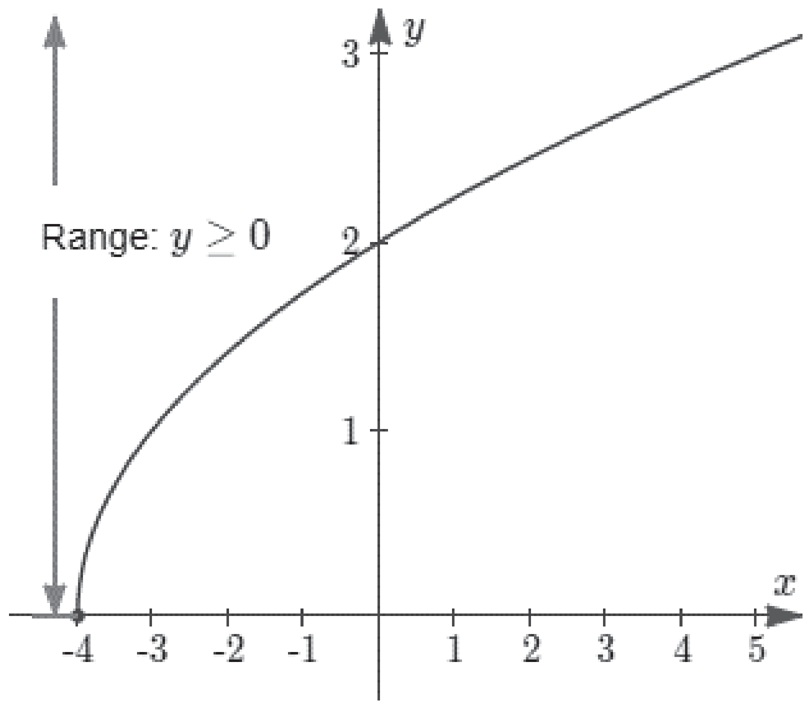
Functions assign outputs to inputs. The domain of a function is the set of all possible inputs for the function. For example, the domain of $f(x)=x^2$ is all real numbers, and the domain of $g(x)=1/x$ is all real numbers except for $x=0$. We can also define special functions whose domains are more limited.

- **Thus, the domain of a function is the set of all values the independent variable can take.**
- **The domain can be specified explicitly or implicitly. When it is implicit, the domain is the set of all real numbers for which the function makes sense.**

Let us take an example:

$$f(x) = \sqrt{x+4}$$

If we plot a graph it will look like the one given below:



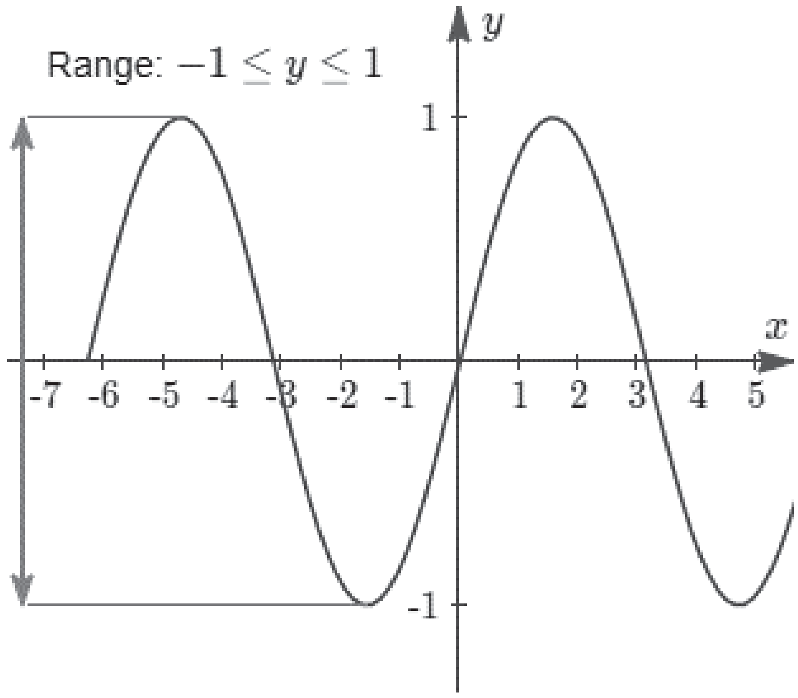
The domain of this function is $x \geq -4$, since x cannot be less than -4. If we try to put value less than -4 we cannot calculate the square root.

The range of a function is the complete set of all possible resulting values of the dependent variable after we have substituted the domain.

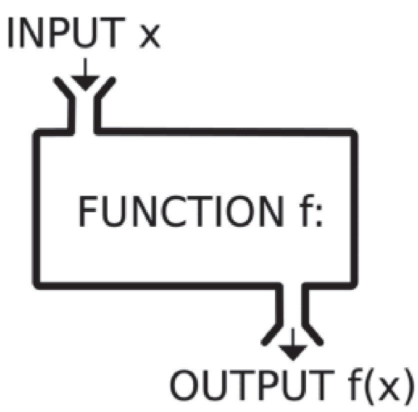
We notice that the curve is either on or above the horizontal axis. For $x \geq -4$, we will always get a zero or positive value of y. We say the **range** in this case is

$y \geq 0$.

Similarly for $y = \sin x$, we can see that range to be between -1 and 1.



A function can be considered as a machine that processes elements of its domain to produce elements of its range. When an element x of domain is fed into input of the machine, the element f(x) of the range comes out in the output as illustrated in the following diagram:



Check your Progress 3 :

a) Click on the following link of a GeoGebra applet to understand the concept of range and domain of a function :
https://www.geogebra.org/m/VGCbyDfr

b) Plot a graph using a spreadsheet and find out the range of the following functions :

$$f(x) = \cos x \text{ and } f(x) = \tan x.$$

8.5 Types of functions:

There are different types of functions. We will try to explore some functions

a) Polynomial Functions

A function $f : R \to R$ is said to be a polynomial function, if for each $x \in R$.

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$$

where $a_0, a_1, a_2, a_3, \quad , a_n$ are real numbers and n is a non-negative integer. The degree of the polynomial is n if $a_0 \neq 0$.

Example: $f(x) = x^3 - x^2 + \sqrt{2}x + 3$ is a polynomial of degree 3.

$f(x) = x^{3/2} + 2x + 1$ is not a polynomial.

$f(x) = 3x + 1$ is a linear polynomial.

$f(x) = -7x^0$ is polynomial of degree 0.

b) Rational Function:

A function defined by the quotient of two polynomials is called a rational function.

Thus $f(x) = \dfrac{f(x)}{g(x)}, g(x) \neq 0$ is a rational function.

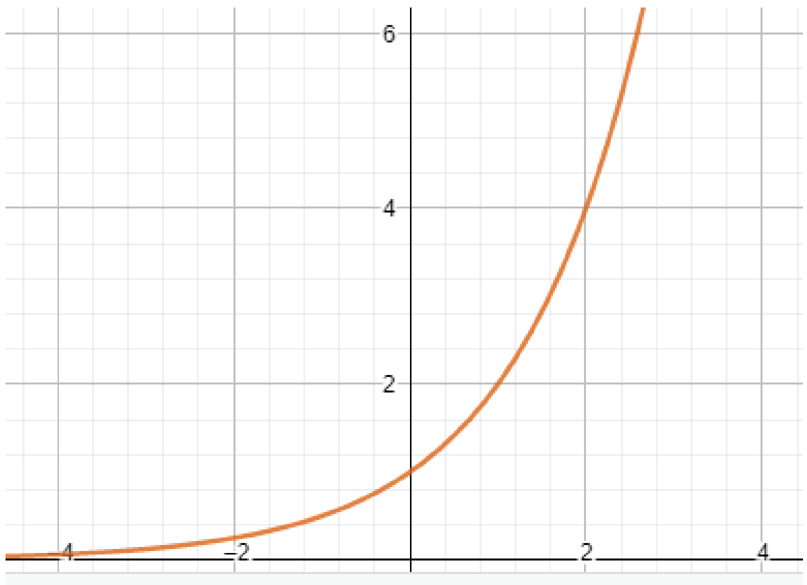
$f(x) = \dfrac{2x^3 + x + 5}{x^2 - 3x}$ is a rational function.

Note: This function is not defined at $x = 0$ and $x = 3$. (Why?)

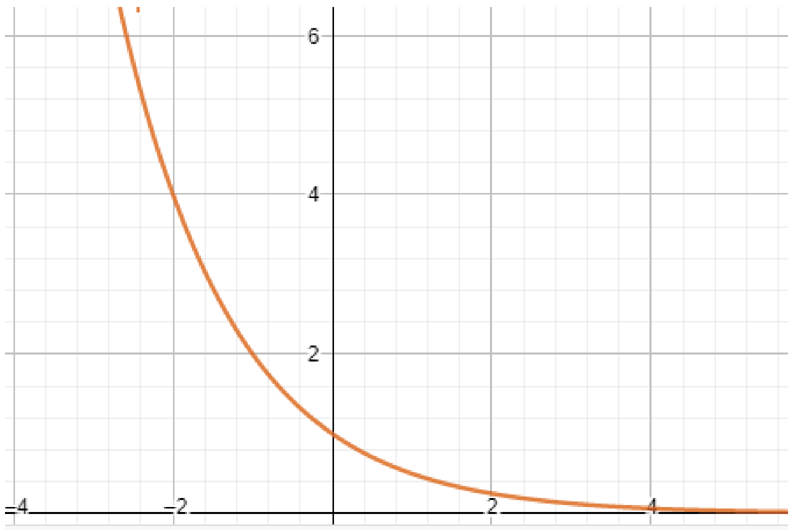
c) **Exponential Function:**

Let x be any real number and $a > 0$ but $a \neq 1$, $f(x) = a^x$ is called exponential function with base a. For different values to the base a, the function $f(x) = a^x$ have different characteristics.

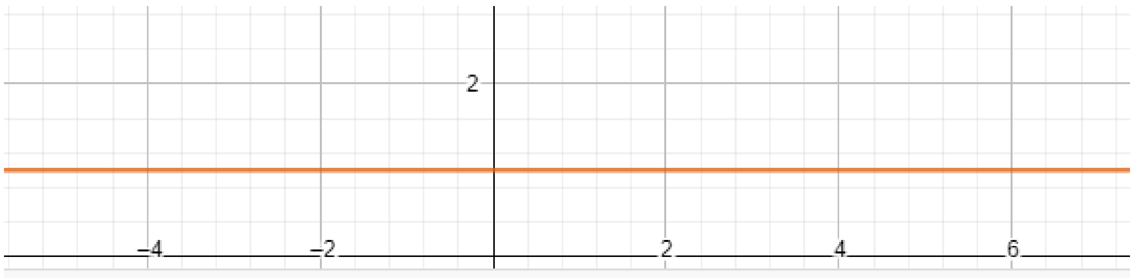
If $a > 1$, $f(x) = a^x$ is strictly increasing.



If $o < a < 1$, $f(x) = a^x$ is strictly decreasing.



If $a = 1$, the graph of $f(x) = a$ is a horizontal line.

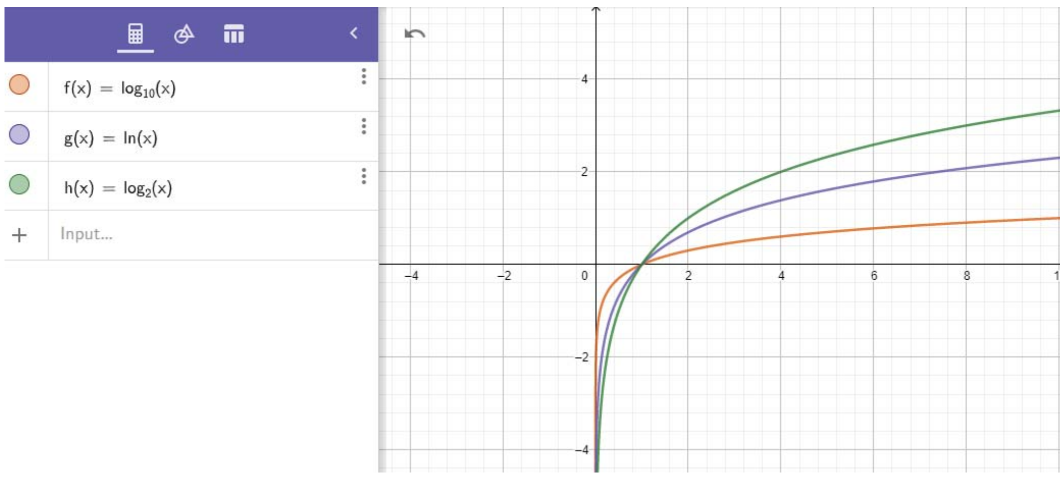


The domain of a^X (with given restrictions on a) is R.

The range of a^X is R i.e., a^X is always positive.

d) **Logarithmic functions :**

These functions are the inverses of exponential functions. The logarithmic function $y = \log_a x$ is defined to be equivalent to the exponential equation $x = a^y$ only under the following conditions: $x = a^y$, $a > 0$, and $a \neq 1$. It is called the logarithmic function with base a. The graphs of logarithmic functions with bases 2, e and 10 are given below.

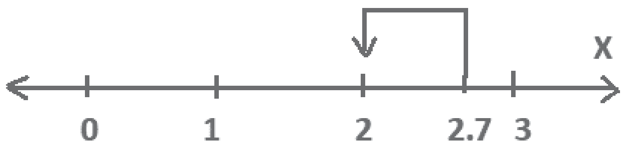


e) **Greatest Integer Function**

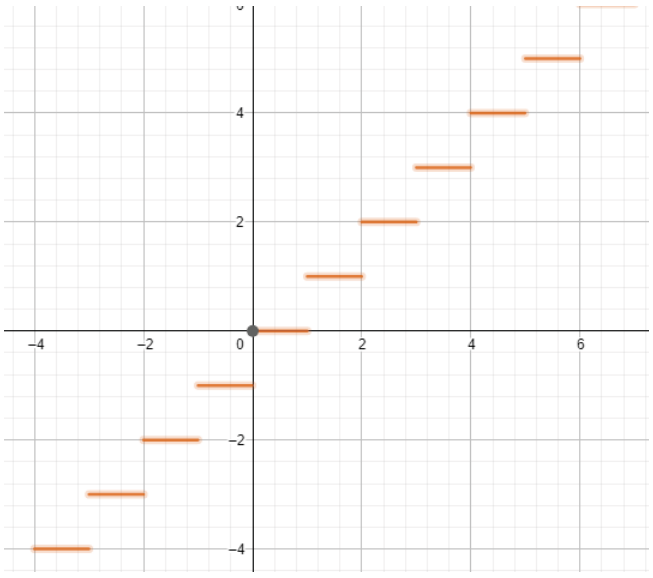
Let x be a real number. The function $f(x) = \lfloor x \rfloor$ where $f(x) = \lfloor x \rfloor$ is the greatest integer $\leq x$ is

called greatest integer function. This is also called step function.

If we examine a number line with the integers and 2.7 plotted on it, we see $\lfloor x \rfloor$



If we plot f(x) = ⌊ x ⌋ we get the following graph

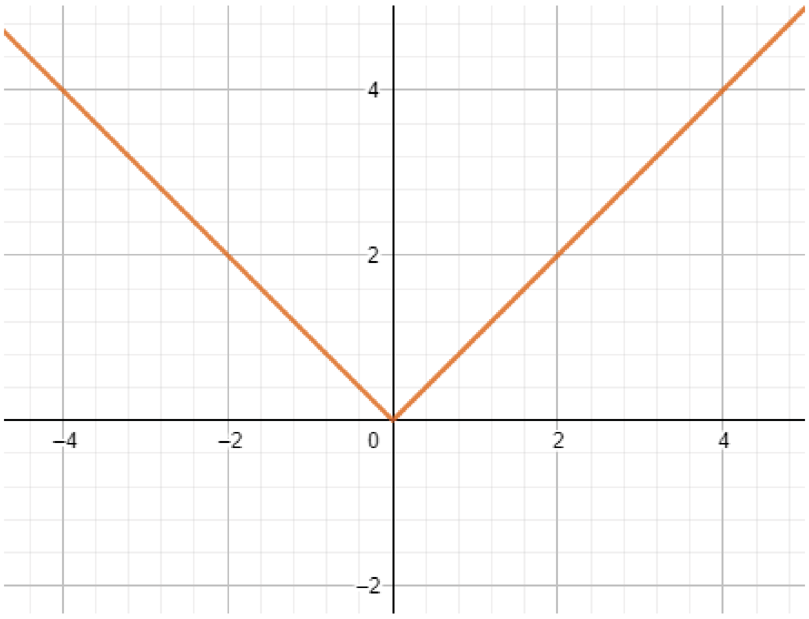


f) **Modulus Function**

For each non-negative value of x, f(x) is equal to x. But for negative values of x, the value of f(x) is the negative of the value of x.

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

It is also called the **absolute value function.** Graph of I x I is given below:

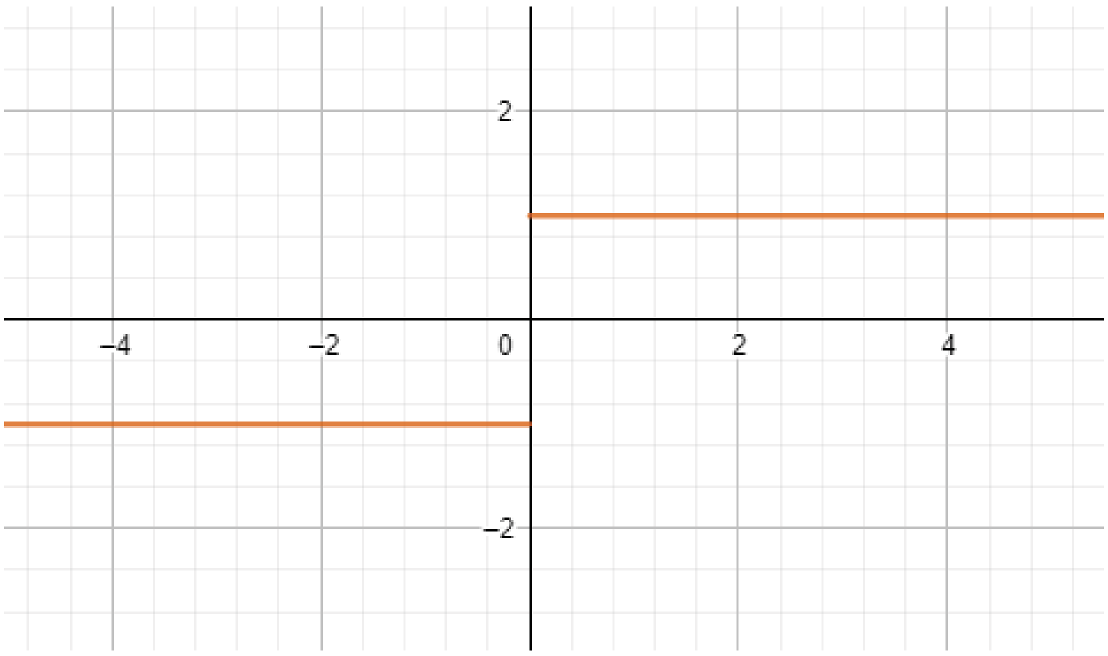


g) **Signum Function**

The signum of a real number, also called sgn or signum, is -1 for a negative number (i.e., one with a minus sign "), 0 for the number zero, or +1 for a positive number (i.e., one with a plus sign"). In other words, for any real x

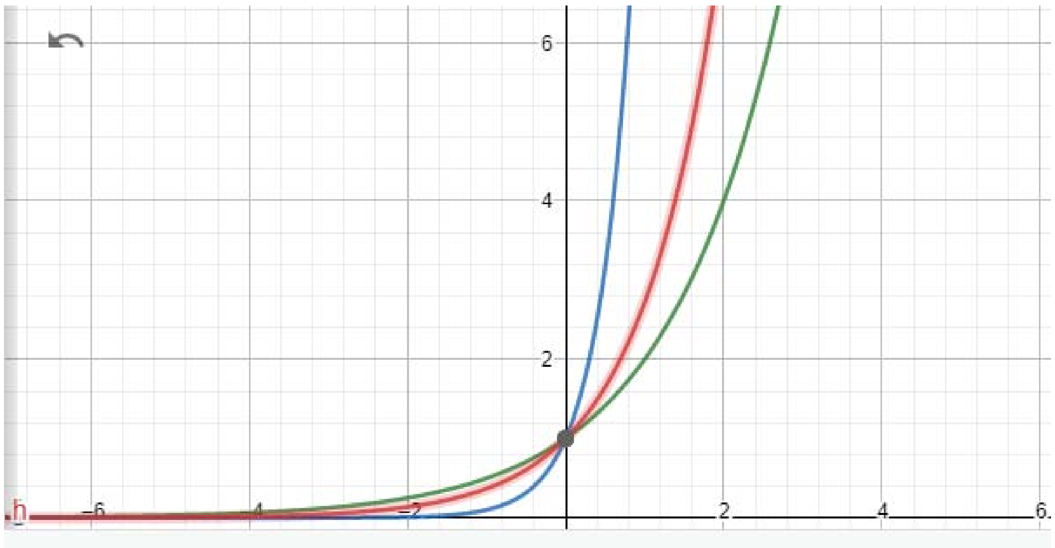
$$\text{sgn}(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

Plot of signum function is given below:



Check your Progress 4:

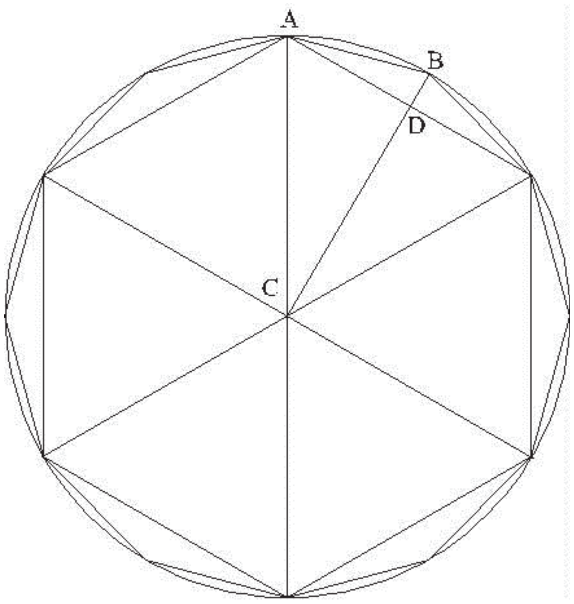
Following are the graphs of h (x) = e^x, g(x) = 10^x and f(x) = 2^x plotted using Geo Gebra graphing calculator. Identify the colour of the graph corresponding to each function.



8.6 Limit of a function:

For understanding Calculus, we need to understand the concept of limit or limiting process. The concept of limit was used to understand the area of a circle in earlier days. The area enclosed by a circle of radius r is given by πr^2.

We can view the circle as a limit of a sequence of regular polygons as shown above:

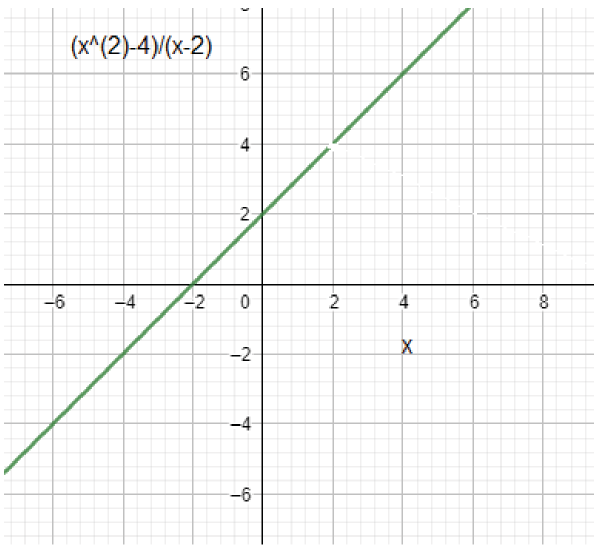


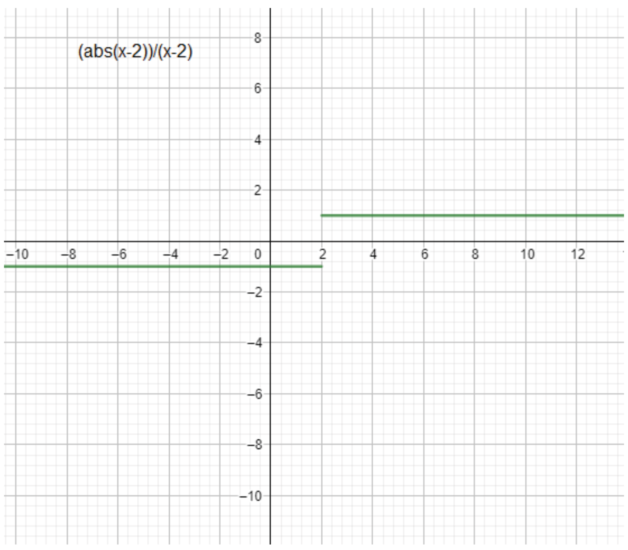
Area of a regular polygon can be obtained by multiplication of half of its perimeter with the distance from the centre to its sides.

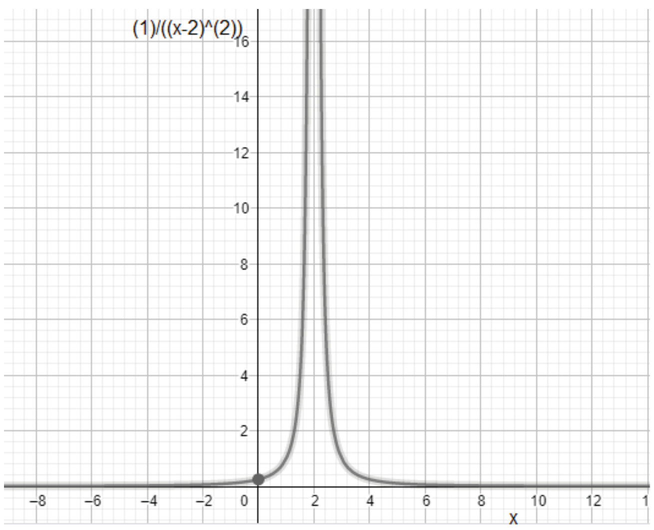
Thus area of circle can be calculated as below if the side of the polygon tends to zero (limiting case).

$A = 1/\boxed{2} \times 2\pi r \times r = \pi r^2.$

Let us understand the concept of limit in case of a function in intuitive way before the formal definition. Note the behavior of each graph at x=2







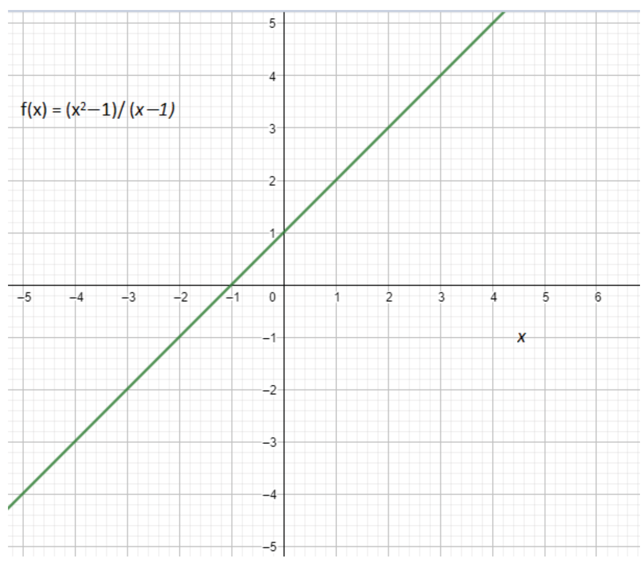
$$f(x) = \frac{x^2 - 4}{x - 2} \qquad g(x) = \frac{|x-2|}{x-2} \qquad h(x) = \frac{1}{(x-2)^2}$$

(a) (b) (c)

We can easily find that these functions are undefined at x=2. Such a statement does not give us a complete picture of the function at this value. To explain the behaviour of graph near x=2 we need to introduce the concept of limit.

Let us consider another function : f(x) = $(x^2 - 1)/(x - 1)$. When we put the value of x= 1 we find that the function is undefined. Now let us put some values other than x = 1 and find out the value of the function as given below.

X	$(x^2 - 1)/(x - 1)$
0.5	1.50000
0.9	1.90000
0.99	1.99000
0.999	1.99900
0.9999	1.99990
0.99999	1.99999
...	...
1.0001	2.00010
1.001	2.00100
1.01	2.01000
1.1	2.10000
1.5	2.50000



It can be observed that as we get close to 1 we get the value of function close to 2. It means that though the function at x=1 is not defined as we get the value close to 1 we get the value of the function approaches 2. In Mathematics we use a concept of limit to express this situation. This result can be stated as :

The **limit** of $(x^2-1)/(x-1)$ as x approaches 1 is **2**

Symbolically same can be written as : $\lim_{x\to 1}\frac{x^2-1}{x-1}=2$

Illustrations:

Let us find out the limit of some functions:

1. Find the limit of $f(x)=4x$, as x approaches 3.

   Substitute the value of x and we get f(3) = 4x3 =12

2. Let us now find out limit of another function as x tends to zero $\lim_{x\to 0}\frac{6x^2-7x}{x}$

   When we substitute x =0 we do not get a definite result.

   Let us simplify and see what happens.

   $$\frac{6x^2-7x}{x}=\frac{x(6x-7)}{x}=6x-7$$

   Now when we put the value of x=0 we get

   $$\lim_{x\to 0}(6x^2-7)=-7$$

Algebra of Limits:

The limit of a sum is the sum of the limits.	$\lim\limits_{x \to a} (f+g)(x) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$ if both limits on the right-hand side exist and are finite.
The limit of a product is the product of the limits.	$\lim\limits_{x \to a} (f.g)(x) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$ if both limits on the right-hand side exist and are finite.
The limit of a quotient is the quotient of the limits.	$\lim\limits_{x \to a} (f/g)(x) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$ if both limits on the right-hand side exist and are finite and the limit in the denominator is not zero

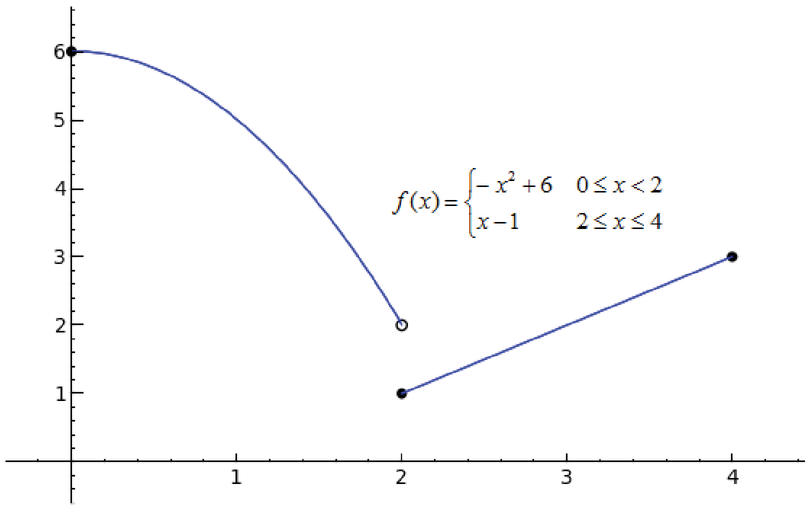
Check your progress: 5

a) Find : $\lim\limits_{x \to 2} (8 - 3x + 12x^2)$

b) Find: $\lim\limits_{x \to 8} \dfrac{2z^2 - 17z + 8}{8 - z}$

8.7 Continuity of a function:

In the case of f(x) = $(x^2 - 1)/(x - 1)$ discussed earlier, irrespective of value of x approaches to 1 from a value less than one or greater than one, the value of the limit of the function is 2. We may not always get a unique value in these two cases. See the above example.



We can say that the limit of f(x) as x approaches 2 from the left is 2, and the limit of f(x) as x approaches 2 from the right is 1. This can be written as:

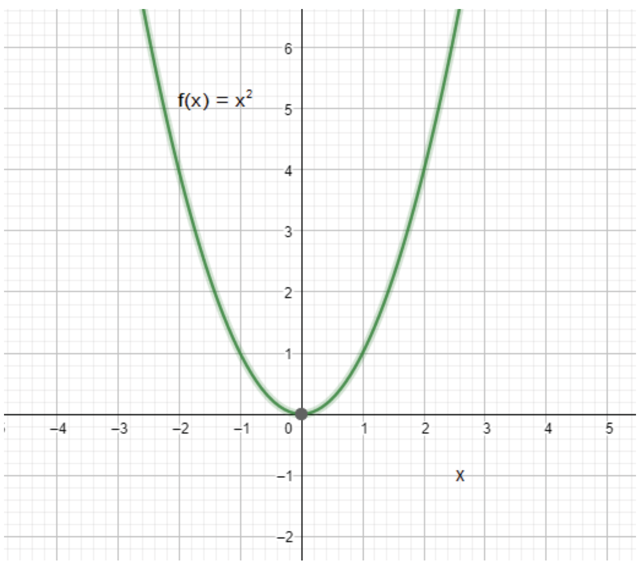
$$\lim_{x\to 2^-} f(x) = 2$$

$$\lim_{x\to 2^+} f(x) = 1$$

The (-) sign indicates "from the left", and (+) sign indicates "from the right".

The function is said to be **discontinuous.** In other words we can say that limit of this function does not exist at x=2.

On the other hand, for the function f(x) = x^2, it can be easily verified that at any point, x=a, lim f(x) = f(a) and the graph of the function is smooth curve having no break anywhere.

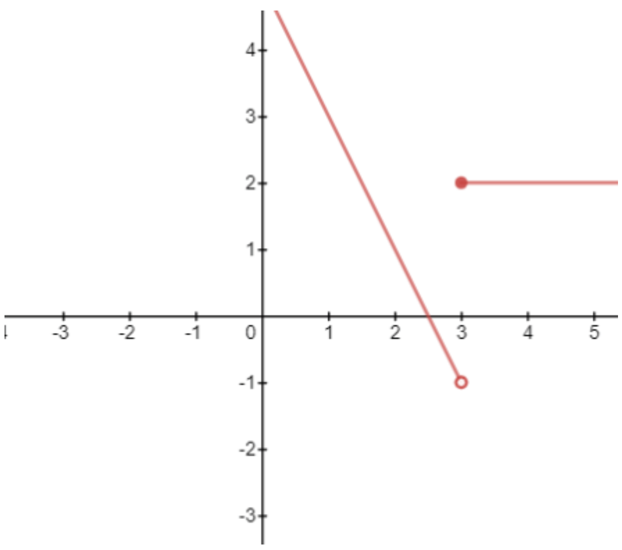


The function $f(x)$ is said to be continuous at the point $x = a$ if its finite value at $x = a$ is equal to the limiting value of $f(x)$ at the point $x = a$. Thus for continuity at $x = a$

Left hand limit = right hand limit = value of the function

Check your progress 6:

a) Study the graph given below:



(i) What y-value is the function approaching as x approaches 3 from the left?

(ii) What y-value is the function approaching as x approaches 3 from the right?

(iii) What (if any) is the actual y-value at x=3? What can you conclude about the function?

b) Following are the examples of Some Continuous Functions. Reflect anc discuss with your friends.

(i) A constant function $f(x) = c$ is continuous everywhere.

(ii) Function $f(x)$ nxn; $n \in N$ is continuous on R.

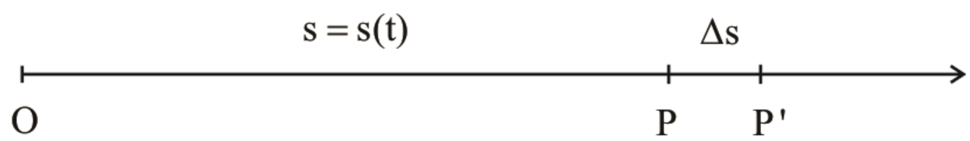
(iii) $\sin x$, $\cos x$ are continuous functions on R.

(iv) $f(x) = |x|$ is continuous function on R.

(v) Polynomial functions are always continuous.

8.8 Instantaneous rate of change and derivative:

One of the important applications of the Calculus is the motion of a particle along a straight line.



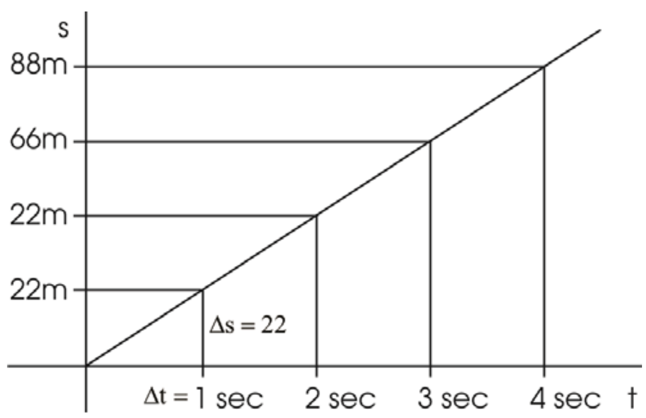
Consider a particle moving in a straight line from a fixed point O to a given point P, and let t be the time elapsed. Then to each value of t there will correspond a distance s, which will be a function of t:

$s = s(t)$.

If the particle moves with a constant velocity of 22 m/s, we can write the distance covered

$s = 22t$

If we plot the function s(t) with respect of t we get the following graph:

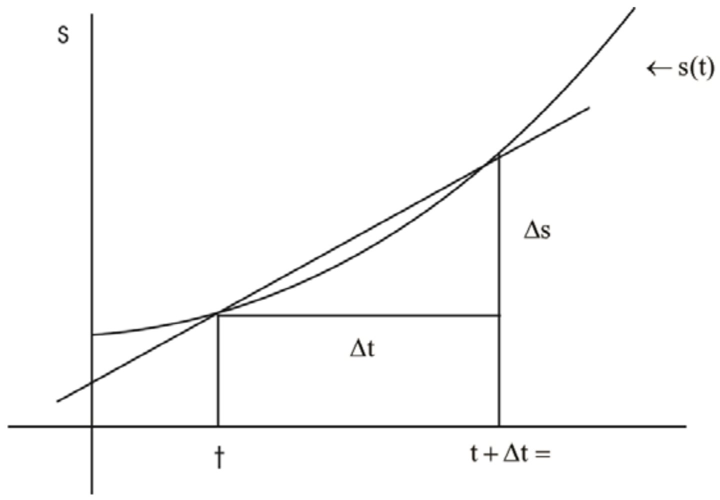


Velocity, which is rate of change of s with respect of t can be calculated by finding out the slope of the graph

$$\frac{\Delta s}{\Delta t} = 22 \text{ meters per second}$$

We can easily see that the value of the slope is constant as velocity is constant in this case.

But in the following case where the velocity of the object is not constant. One can find out average value of velocity over a period of time. But how can we find out the velocity of an object at a particular instant t or when $\Delta t = 0$



You may wonder how can we find the value of velocity when $\Delta t = 0$. Here the concept of limit will help us in finding out a definite value of this instantaneous velocity. We can define instantaneous velocity using slope as the limiting value of $\Delta s/ \Delta t$ when Δt tends to zero. This can be written as :

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

It can also be said to be the value of slope of the tangent line at t.

Thus instantaneous velocity is said to be derivative of s with respect to t . **Differentiation** is the action of computing a derivative.

The derivative of a function y = f(x) of a variable x is a measure of the rate at which the value y of the function changes with respect to the change of the variable x. It is called the derivative of f with respect to x. If x and y are real numbers, and if the graph of f is plotted against x, the derivative is the slope of the tangent to this graph at each point.

For example derivative of the function f at a can be written as :

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

When the limit exists, f is said to be differentiable at a. Here f′(a) is one of several common notations for the derivative If f(x) is a function of x derivative can also be written as: $\dfrac{df}{dx}$

If we use this definition to obtain the derivative then we are using first principle to find out derivative.

Illustration : Let us find out derivative of $y = 2x^2 + 3x$. using first principle:

$$\begin{aligned} f(x) &= 2x^2 + 3x \text{ so} \\ f(x+h) &= 2(x+h)^2 + 3(x+h) \\ &= 2(x^2 + 2xh + h^2) + (3x + 3h) \\ &= 2x^2 + 4xh + 2h^2 + 3x + 3h \end{aligned}$$

We now need to find

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[2x^2 + 4xh + 2h^2 + 3x + 3h\right] - \left[2x^2 + 3x\right]}{h}$$

$$= \lim_{h \to 0} \frac{4xh + 2h2 + 3h}{h}$$

$$= \lim_{h \to 0} (4x + 2h + 3)$$

$$= 4x + 3$$

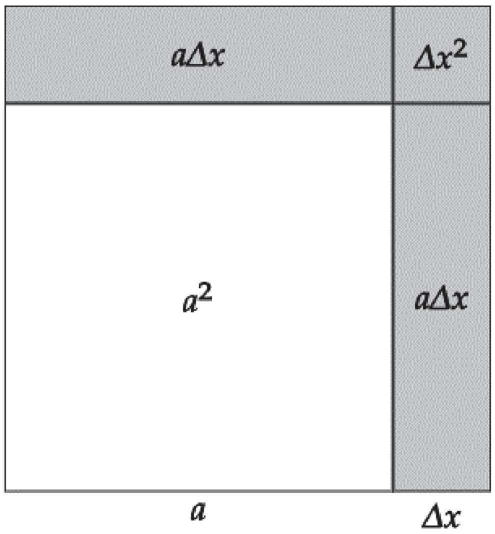
It will be worthwhile to know derivatives of certain functions.

Common Functions	**Function**	**Derivative**
Constant	c	0
Line	x	1
	ax	a
Square	x^2	2x
Square Root	√x	$(½)x^{-½}$
Exponential	e^x	e^x
	a^x	$\ln(a)\ a^x$
Logarithms	ln(x)	1/x
	$\log_a(x)$	1 / (x ln(a))

Check your Progress 7:

a) Show that the derivative of a constant is zero and derivate of ax with respect to x is a.

b) Let function $y=x^2$ that measures the area of a metallic square of side x. If at any given time the side of the square is a, and we heat the square uniformly increasing the side, what is the tendency of change of the area in that moment?



Some rules which will be of help in finding derivatives are:

Rules	**Function**	**Derivative**
Multiplication by constant	cf	cf′
Power Rule	x^n	nx^{n-1}
Sum Rule	f + g	f′ + g′
Difference Rule	f - g	f′ – g′
Product Rule	fg	f g′ + f′ g
Quotient Rule	f/g	$(f'\,g - g'\,f)/g^2$
Reciprocal Rule	1/f	$-f'/f^2$
Trigonometry (x is radians)	sin(x)	cos(x)
	cos(x)	–sin x
	tan(x)	$\sec^2 x$

8.9 Derivatives of algebraic functions using chain rule :

A function is *composite* if you can write it as *f*(*g*(*x*)). In other words, it is a function within a function, or a function of a function. For example $(5x-2)^3$ is composite function. Here 5x -2 can also be considered as a function. Chain rule helps us in finding the derivative of a composite function.

If f is a composite function then derivative of f with respect x can be written using chain rule as :

$$\frac{d}{x}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right)g'(x)$$

Where g is the inner function and f is the outer function.

Let us find out the derivative $(5x-2)^3$ using chain rule

$(5x-2)^3$ is made up of g^3 and 5x-2:

- $f(g) = g^3$
- $g(x) = 5x–2$

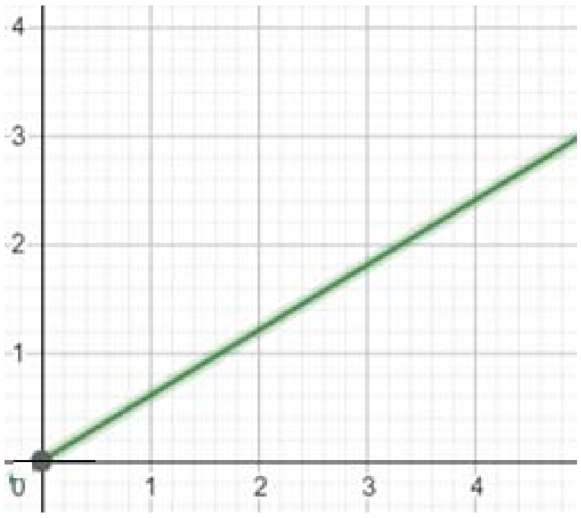
The individual derivatives are:

$f'(g) = 3g^2$ (by the Power Rule)

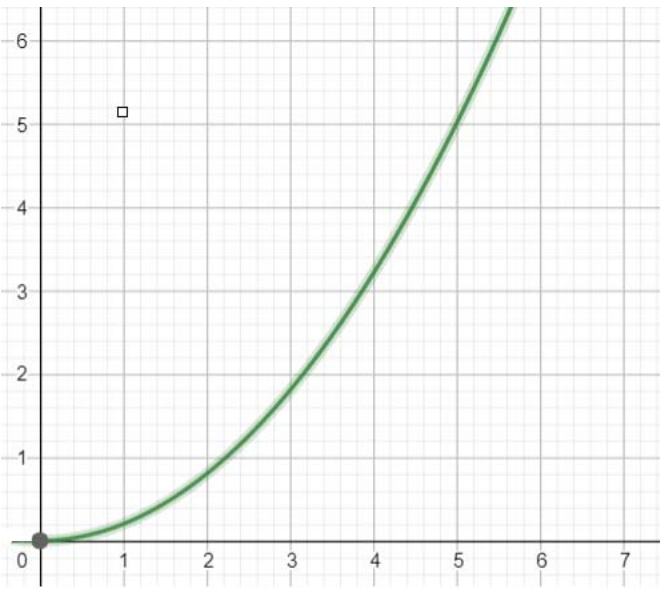
$g'(x) = 5$

Therefore derivative $(5x-2)^3 = 3\ (5x-2)^2 x5\ = 15\ (5x-2)^2$

One way to interpret the derivative f′ is to say that f′(k) at k is the rate of change of f at x =k Let us say that a water tank is filled at constant rate and the volume of water in the tank is a function of time given by V (t) =3/5 t . If we plot a graph of V(t) with respect to t we get the following :



Here 3/5 (which is a constant) is the slope of the graph and can be determined by finding the derivative of V(t) with respect to t. That means the water tank is being filled at the rate of 3/5 liters per second. But the situation may not be that simple. If the volume of water being filled in the tank is non linear and is given by the $V(t) = 0.2\ t^2$. Here we will not have constant rate of change of volume. Let us plot the graph and see.



We can find instantaneous rate of change of volume of water in the tank by finding the derivative of V(t), which comes out to be 0.2x2t = 0.4t. This value changes with change in time (t). If we want to find the rate of change of volume with respect to time at any instant we just substitute the value. For example, the rate of change will with 2 L/s at t =5 second

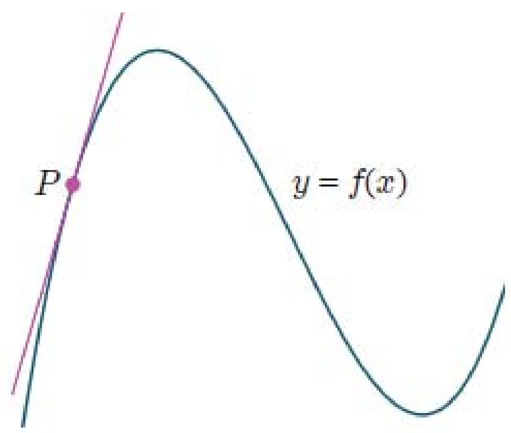
Check your Progress 8 :

a) Find the rate of change of the area of a circle with respect to its radius r when r = 5 cm.

b) On heating, the volume of a metal cube is increasing at a rate of 9 cubic centimeters per second. How fast is the surface area increasing when the length of an edge is 10 centimeters?

8.10 Tangent line and equation of tangent :

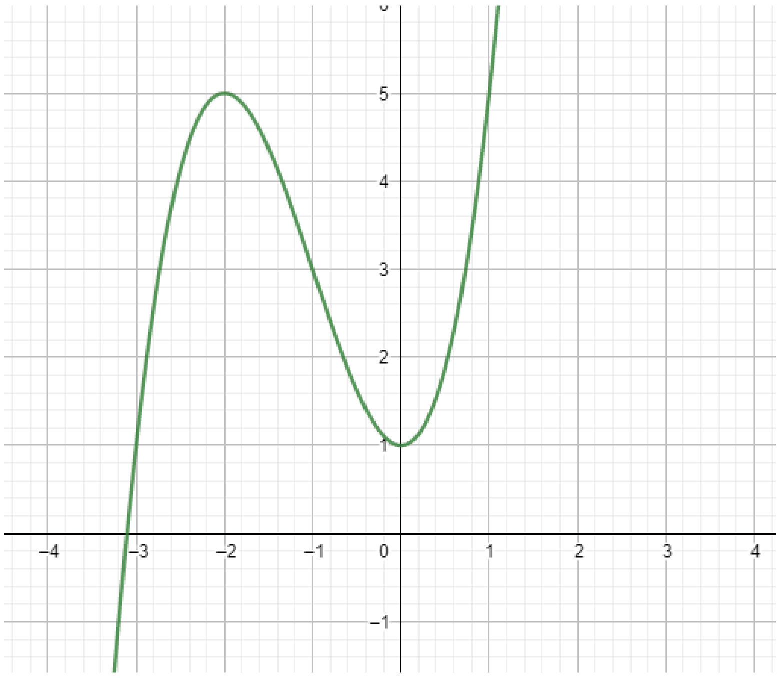
A tangent line (or simply tangent) to a plane curve at a given point is a straight line that just touches the curve at that point. More precisely a straight line is said to be a tangent of a curve y = f(x) at point x = c if the line passes through the point (c, f (c)) on the curve and has a slope f′(c) where f′ is the derivative of f. In the following diagram a tangent is drawn at the point P.



When we say tangent line for a function can be found by computing the derivative, it only means that " the derivative measures the slope of the tangent line". Following step can be used to find the slop of the a tangent line for a function f(x) at a given point x = a.

1. Find the derivative f′(x)
2. Put the value of x =a to get the slope. i.e slope m = f′(a)
3. Find out the y coordinate of the point which is b = f(a)

Use the formula y = m(x-a) +b and put values of m,a and b to get the equation of tangent line at point (a,b) Let us take a concrete example to understand this procedure. For a function $f(x) = x^3 + 3x^2 + 1$. We'll find the tangent lines at a few different points. Plot of the graph is as under:



Let us find out the equation of tangent lines at the points x = -3,

Case : 1   x = a = -3

The derivative of $x^3 + 3x^2 + 1$. Is $3x^2 + 6x$

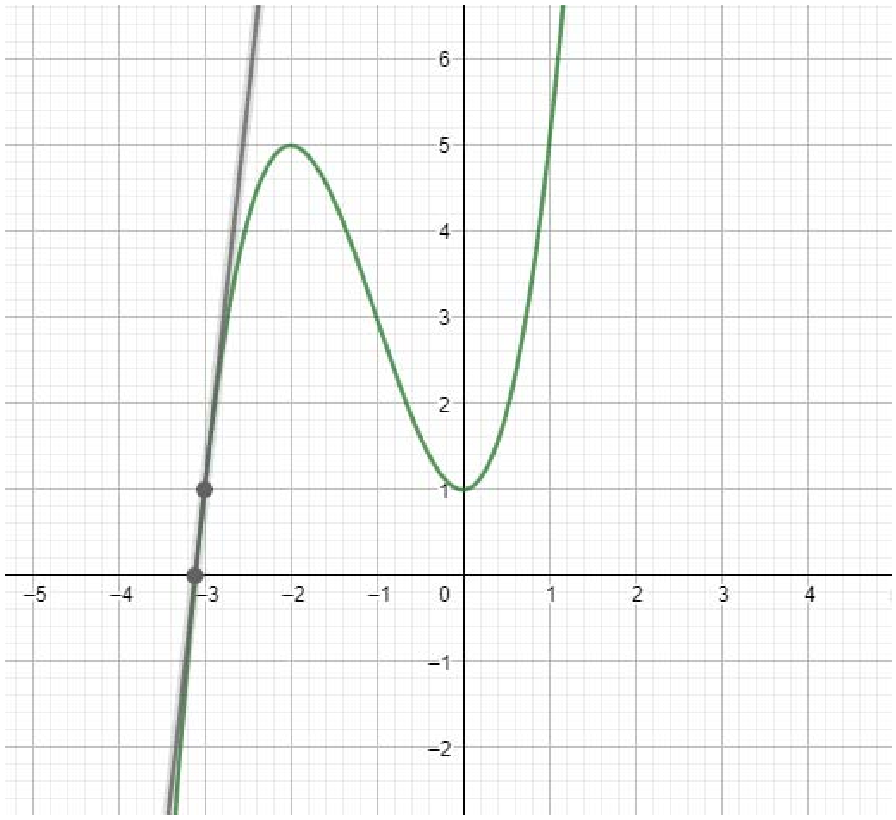
Slope m = f'(a) = 9

If the value of x = a= -3 value of y = b = 1

Thus equation of the tangent line is y =m (x-a)+b

Thus equation of tangent line at point (-3,1) to the function $f(x) = x^3 + 3x^2 + 1$ is y = 9(x+3)+1

If we draw this tangent line it will look like the one given below:



Check your Progress 9:

a. For the function $f(x) = x^3 + 3x^2 + 1$ given above, tangent line at point -3 is drawn using GeoGebra graphing calculator. Draw the tangent line using this application at x= -2 and x=1.

b. Find the equations of a line tangent to $y = x^3-2x^2+x-3$ at the point x=1.

Summary:

- Function, in mathematics is an expression, rule, or law that defines a relationship between one variable (the independent variable) and another variable (the dependent variable).
- In a function, the input value must have one and only one value for the output.
- The domain of a function is the set of all values, the independent variable can take.
- The range of a function is the complete set of all possible

resulting values of the dependent variable after we have substituted the domain.

- Different types of functions are Polynomial function, rational function, Exponential function, Logarithmic function, Greatest integer function, Signum functions etc. All these functions can be plotted on a graph.
- Concept of limit can be used to find out the value of function when it is not defined at a particular value.
- The function f(x) is said to be continuous at the point x = a if its finite value at x = a is equal to the limiting value of f(x) at point x = a. Thus for continuity at x = a, Left hand limit = right hand limit = value of the function
- The derivative of a function y = f(x) of variable x is a measure of the rate at which the value y of the function changes with respect to the change of the variable x. It is called the derivative of f with respect to x.
- If x and y are real numbers, and if the graph of f is plotted against x, the derivative is the slope of the tangent to this graph at each point.
- Derivative of the function f at a can be written as :

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

- If f is a composite function then derivative of f with respect x can be written using chain rule as :

$$\frac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(x)\right)g'(x)$$

Where g is the inner function and f is the outer function.

- The concept of derivative can be used to find the slope of the tangent line for a function f(x) at a given point x = a. Steps are:

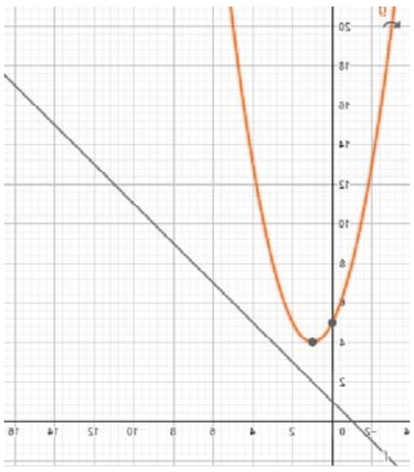
  - Find the derivative f′(x)
  - Put the value of x =a to get the slope. i.e slope m = f′(a)
  - Find out the y coordinate of the point which is b = f(a)
  - Use the formula y = m(x-a) +b and put values of m,a and b to get the equation of tangent line at point (a, b)

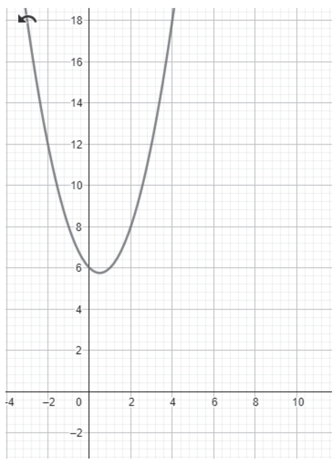
Solutions to Check your Progress :

1 a (i) is not a function as input 1 has two outputs: 0 and 5.

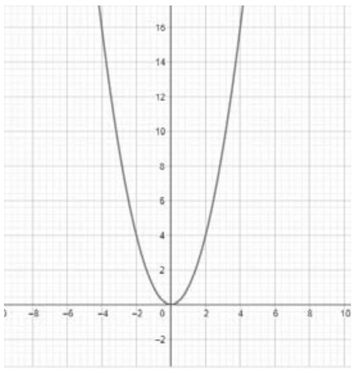
(ii) is a function. Each input has only one output. Same output (4) for same point does not matter.

b) (f+g) (x) = x^2-x+6 and the graphs are as under:

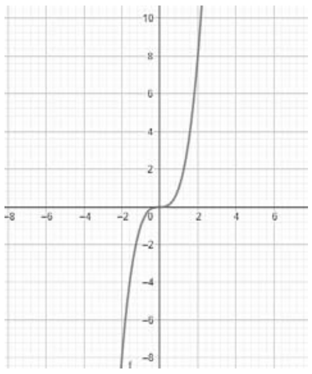




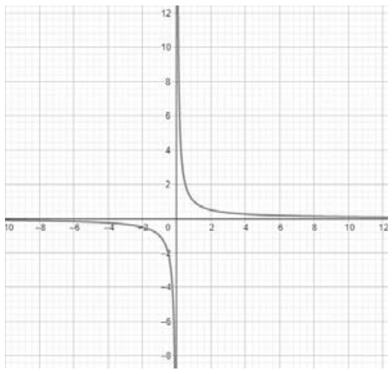
2. Graphs are as under :



a)



b)



c)

3.

	$f(x) = 2^x$
	$g(x) = 10^x$
	$h(x) = e^x$

4. Answers:

a) $\lim\limits_{x \to 2} (8 - 3x + 12x^2) = 8 - 3(2) + 12(4) = \boxed{50}$

b) $\lim\limits_{z \to 8} \dfrac{(2z^2 - 17z + 8)}{8 - z} = \lim\limits_{z \to 8} \dfrac{(2z-1)(z-8)}{-(z-8)} = \lim\limits_{z \to 8} \dfrac{2z-1}{-1} = \boxed{-15}$

5 a)

(i) y = -1
(ii) y=2
(iii) y = 2 , the function is not continuous at x=2

6 b) Rate of change of area with respect to side can be calculated by finding the derivative of x^2 with respect to x and if we find its value of at 'a' we get the tendency of change of the area in that moment. By using first principle we get this value = 2a

7 a) Putting value of r in $2\pi r$ we get $10\pi\ cm^2 / cm$.

b) $3.6\ cm^2/s$

8. b) $y' = 3x^2 - 4x + 1$

At x=1 we get the value of slope = $3(1)^2 -4(1) +1 = 3-4+1= 0$.

Value of y at x =1 is b = $(1)^3-2(1)^2+1-3 = 1-2+1-3 = -3$.

Using the formula : $y = m(x-a) +b = -3$

$y=-3$ is the equation of tangent at $x=1$

Chapter - IX

Probability

Learning Objectives:

Students will be able to:

- Find out sample space in a random experiment
- Recognize whether events are mutually exclusive, independent or dependent
- Apply theorem of total probability in different situations
- Apply Bayes' theorem in practical situations.

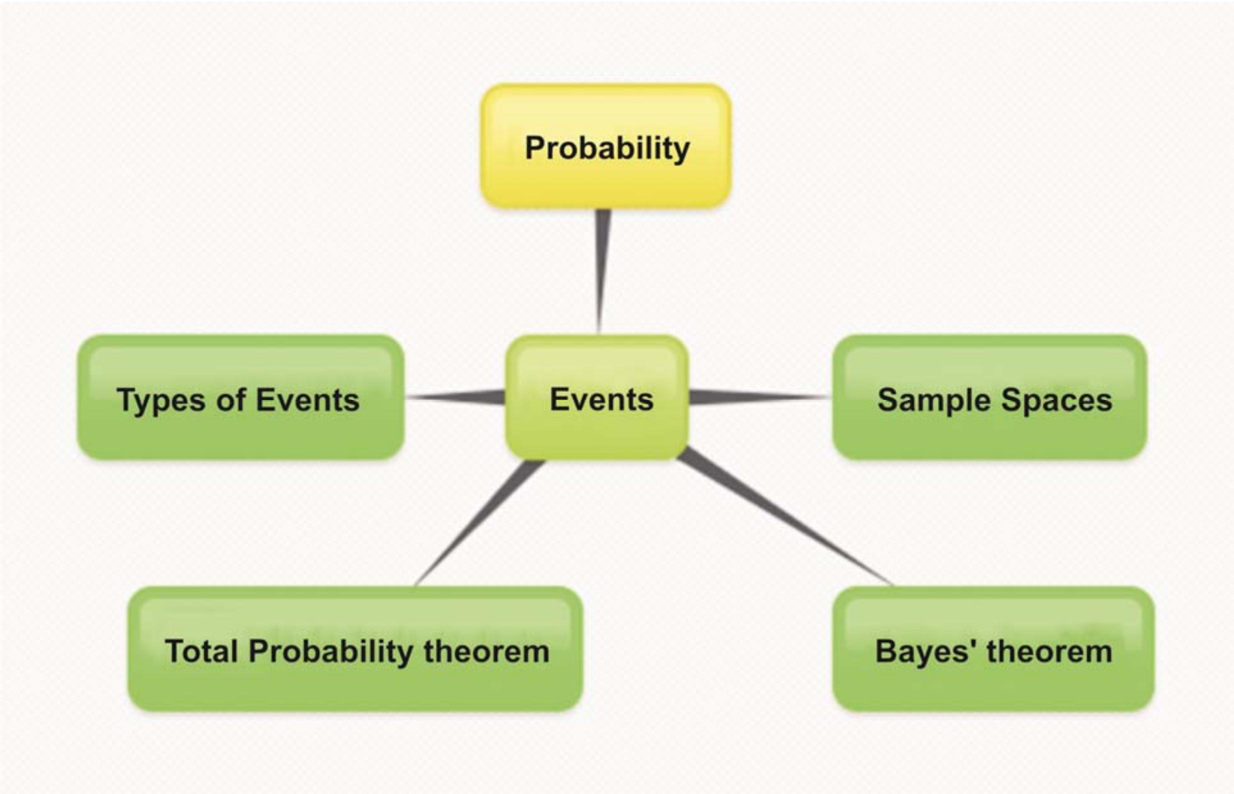
Before you start you should know

Sets, Union and Intersection of sets, Definition of probability and Addition and multiplication rule of probability.

Revise your ideas on probability here:

https://www.khanacademy.org/math/probability/probability-geometry/probability-basics/a/probability-the-basics

Concept Map



9.1 Introduction

Probabilities are part and parcel of life. Some examples from daily life where probability calculations are involved are the determination of premium of insurance, the introduction of new medicines in the market, opinion and exit polls and weather forecasts, In everyday life, many of us use probabilities in our language and say things like "I am one hundred percent sure".

Many important practical problems involving probability can be solved using ***Total Probability Theorem and Bayes' theorem***. These theorems can be used to solve the problems related to insurances, to find out the rate of risk of lending money to potential borrowers, determine the accuracy of medical test results etc. In this unit we are going to discuss more about the practical applications of these two theorems. To understand and apply these two theorems we should know some basic concepts like **random experiment, sample space, events, mutually exclusive events, independent and dependent events.**

9.2 Random experiment and sample space:

An experiment is called random if it has more than one possible outcome and it is not possible to predict the outcome in advance. For example before rolling a die, we do not know the result and hence is random in nature. Rolling a die, therefore is an example of a random experiment.

A **random experiment** is a process by which we observe something uncertain.

After the experiment, the outcome of the random experiment is known. An **outcome** is a result of a random experiment.

The set of all possible outcomes is called the **sample space**.

Sample space can be represented as a **Set** of all possible outcomes. Thus in the context of a random experiment, the sample space is our universal set.

Here are some examples of random experiments and their sample spaces represented in the form of sets.

<table>
<tr><th>Random experiment: Tossing a coin</th><th>Random experiment: Rolling a die</th></tr>
<tr><td>
</td><td></td></tr>
<tr><td>Sample space: S={head, tail}</td><td>Sample space: D={1,2,3,4,5,6}</td></tr>
</table>

Random experiment : Observing the number of android phones sold by a company in 2019.  Sample space: L={0,1,2,3,⋯}	**Random experiment:** Observing the number of goals scored in a soccer match.  Sample space: K={0,1,2,3,⋯}

When a random experiment consists of performance repeated several times, we call each one of them a **trial**. Thus, **a trial is a particular performance of a random experiment.** In tossing a coin, each trial will result in either heads or tails. **Note that the sample space is defined based on how you define the outcome of your random experiment.** For example, if we toss a coin three times and observe the sequence of heads/tails as an outcome, which is usually the understanding, the sample space may be defined as:

S={(H,H,H),(H,H,T),(H,T,H),(T,H,H),(H,T,T),(T,H,T),(T,T,H),(T,T,T)}.

And if the outcome of the experiment of tossing a coin three times is defined as the number of heads, then S = {0,1,2,3}

Example 1 :

In each of the following experiments specify appropriate sample space

(i) A girl has a one rupee coin, a two rupee coin and a five rupee coin in her pocket. She takes out two coins out of her pocket, one after the other.

(ii) A person is noting down the number of accidents along a busy road during a year.

Solution:

(i) Let Q denote a 1 rupee coin, H denote a 2 rupee coin and R denote a 5 rupee coin. The first coin she takes out of her pocket may be any one of the three coins Q, H or R. Corresponding to Q, the second draw may be H or R. So the result of two draws may be QH or QR. Similarly, corresponding to H, the second draw may be Q or R. Therefore, the outcomes may be HQ or HR. Lastly, corresponding to R, the second draw may be H or Q. So, the outcomes may be RH or RQ.

Thus, the sample space is S={QH, QR, HQ, HR, RH, RQ}

(ii) The number of accidents along a busy road during the year of observation can be either 0 (for no accident) or 1 or 2, or some other positive integer.

Thus, a sample space associated with this experiment is S= {0,1,2,...}

Example-2 : A bag has 3 blue and 4 white balls in it. If a tossed coin shows head, we draw a ball from the bag. If it shows tail we throw a die to show a number. Describe the sample space of this experiment.

Solution: Let us denote blue balls by B1, B2, B3 and the white balls by W1, W2, W3, W4.

Then a sample space of the experiment is

S = { HB_1, HB_2, HB_3, HW_1, HW_2, HW_3, HW_4, T1, T2, T3, T4, T5, T6}.

Here HB_1 means head on the coin and ball B_1 HW_1 means head on the coin and ball W_1. For any number i, HB_i means head on the coin and ball B_i. HW_i means head on the coin and ball W_i. Similarly, Tj means tail on the coin and the number j on the die. Here, i = 1, 2, 3; j = 1, 2, 3, 4, 5, 6

Check your Progress - 1 :

1. Write down an experiment in practical life whose sample space is $S = \{0,1,2,...\}$

2. Suppose 3 bulbs are selected at random from a lot of bulbs. Each bulb is tested and classified as defective (D) or non – defective (N). Write the sample space of this Experiment.

9.3 Event and its probability:

When we say "Event" associated with a random experiment we mean a set of one (or more) outcomes. We have seen earlier that sample space of tossing a coin three times is defined as.

$$S=\{(H,H,H),(H,H,T),(H,T,H),(T,H,H),(H,T,T),(T,H,T),(T,T,H),(T,T,T)\}.$$

Suppose, we want to find only the outcomes which have at least two heads; then the set of all such possibilities (outcomes) can be given as:

$$E=\{(H,H,H),(H,H,T),(H,T,H),(T,H,H)\}.$$

E is an event associated with this experiment. Thus, an event is a subset of the sample space, i.e., E is a subset of S. Even the empty set is a subset of S, hence an event, called the Impossible event. The S, itself is a subset of S, hence an event, called 'Sure' event.

There could be a lot of events associated with a Random Experiment. For any event E to occur, the outcome of the experiment must be an element of the set E. **Then how to calculate the probability of an event?**

The ratio of number of favourable outcomes to the total number of outcomes is defined as the probability of occurrence of an event.

Note: This definition applies when the outcomes of the experiment are equally likely to occur.

So, the probability that an event will occur is given as:

$$P(E) = \frac{\text{Number of Favourable Outcomes}}{\text{Total Number of Outcomes}}$$

Consider the event A = {2, 4, 6, 8} associated with the experiment of drawing a card from a deck of ten cards numbered from 1 to 10. Clearly the sample space is S = {1, 2, 3, ...,10}

If all the outcomes 1, 2, ...,10 are considered to be equally likely, then the probability **of each outcome is** $\frac{1}{10}$

Now the probability of event A is

P(A) = P(2) + P(4) + P(6) + P(8)

(Following an axiom of Probability theory)

$$= \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$$

We have calculated the probability of an event as sum of probabilities of each outcome in that event. While calculating the probability of each outcome we have taken into account sample space for finding the total number of outcomes.

Similarly we can also find an event which is **not A**. Let us write down the event not A with reference to the example we have discussed earlier.

A′ = {1, 3, 5, 7, 9, 10}

P(A′) = P(1) + P(3) + P(5) + P(7) + P(9) + P(10) = $6 \times \frac{1}{10} = \frac{6}{10} = \frac{3}{5}$

We can easily see that **P(A') = 1- P(A)**

This means that , if the event A occurs event '*not A*' cannot occur. They are called mutually exclusive events.

Note : Two mutually exclusive events need not be compliment of each other.

Two events are said to be mutually exclusive if the occurrence of one rules out the occurrence of the other. E and F are mutually exclusive if $E \cap F = \varnothing$

Some terms Similar to this term related to events are used in our future discussions. Let us understand these terms.

Independent Events and Dependent Events:

If the probability of occurrence of an event is completely unaffected by the occurrence (or non-occurrence) of another event, such events are termed as independent events. The events that are not independent are called **dependent events.**

Impossible and Sure Events:

If the probability of occurrence of an event is zero such an event is called an impossible event. If the probability of occurrence of an event is 1, it is called a sure event. In other words, the empty set ϕ represents an impossible event and the sample space S represents a sure event.

Exhaustive Events:

A set of events is called exhaustive if all the events together consume the entire sample space. Their union is the sample space.

Mutually Exclusive Events:

If the occurrence of one event excludes the simultaneous occurrence of another event, then such events are called mutually exclusive events. For example, if $S = \{1, 2, 3, 4, 5, 6\}$ and E_1, E_2 are two events such that E_1 consists of numbers less than 4 and E_2 consists of a number greater than 5.

So, $E_1 = \{1,2,3\}$ and $E_2 = \{6\}$.

Then, E_1 and E_2 are also called mutually exclusive events.

Dependent Events

Mr. Verma and Ms. Banerjee plan to invest some money in the shares of one of the companies in the following sectors:

1. Banking
2. Software development
3. Metal
4. Automobiles

They approach the same financial analyst for advice and his recommendation has definite influence on their decision.

The two events

E = Mr. Verma invests in shares of banking sector

F = Ms. Banerjee invests in metal stocks

are dependent events as they have the same financial advisor influencing their decision.

Independent Events

A missile radar detective system consists of two radar screens 'RSO2A' and 'RQ54L'. The probability that these will detect an incoming missile are 0.9 and 0.85 respectively.

The two events

E = radar screen 'RS02A' detects the missile

F = radar screen 'RQ54L' detects the missile

are independent events as the probability of detection by any one is not affected by the detection of missile by of the other.

Impossible and Sure Events

Impossible Event

The cut off list for centralized admission to B.Com program in various colleges is as under:

College	Minimum Percentage
Hindu College	95%
St. Stephens	97%
Hansraj	95%
SRCC	96%

If Sarthak who scored 92% in class 12 has applied for admission in B.Com program then the event

E = Sarthak gets admission in one of these colleges

is an impossible event as his marks are less than the minimum cut off required

Sure Event

A manufacturer of earphone claims that not more than 2 pieces of every 1000 earphones manufactured are defective. It also states that a consignment of earphones for export is rejected if it contains atleast 3 defective pieces per 1000.

The event:

A = The consignment for export by the manufacturer is accepted is a sure event.

Exhaustive Events

An inhabitant is selected at random from a city.

The events

S_1 = The person selected is covid +ve

S_1'= The person selected is not covid +ve

S_2 = The person is diabetic

The events S_1 and $S_1'$are exhaustive events as they cover the whole sample space.

Mutually Exclusive Events

A packet containing flower seeds contains 30 seeds of which 12 seeds are of red flower, 8 seeds of yellows flowers and 10 seeds of blue flowers.

The two events

E_1 = the seed produces red flower

E_2 = the seed produces yellow flower

are mutually exclusive events as the flower can be either red or yellow in colour.

Mutually Exclusive and Exhaustive Events

A financial institution sanctioned the following type of loans in 2019.

	Type of loans	Number of Approvals
1.	Business loan	18
2.	Debit consolidation loan	15
3.	Economic opportunity loan	17

If an application from approved loans is selected at random for review then the events:

E_1 = application for business loan

E_2 = application for debit consolidation loan

E_3 = application for economic opportunity loan

are mutually exclusive and exhaustive events as E_1, E_2 and E_3 have nothing in common and all the applications are of one of these types.

Check your progress-2:

Give real life examples of the following:

1. Independent Events and Dependent Events
2. Impossible and Sure Events
3. Exhaustive Events
4. Mutually Exclusive Events

Mutually exclusive and exhaustive Events

9.4 Conditional Probability

If we have two events associated with the same experiment, does the information about the occurrence of one of the events affect the probability of the other event? Let us try to answer this question by taking up a random experiment in which the outcomes are equally likely to occur.

Consider the experiment of tossing three fair coins. The sample space of the experiment is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Since the coins are fair, we can assign the probability $\frac{1}{8}$ to each sample point

Let us consider two events E & F. Let E be the event where 'at least two heads appear' and F be the event where 'first coin shows tail'.

Now, suppose we are given that the first coin shows tail, i.e. F occurs, then what is the probability of occurrence of E?

In the above example above we can clearly see that we have placed a condition: When F occurs, then what is the probability of occurrence of E?

Do you think the probability of event E with the condition will be same or different?

Let E be the event 'at least two heads appear' and F be the event 'first coin shows tail'.

Then we can write the sample spaces as:

E = {HHH, HHT, HTH, THH} and F = {THH, THT, TTH, TTT}

$\therefore$ P(E) = P (HHH) + P (HHT) + P (HTH) + P (THH)

$$=\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$$

P(F) = P ({THH}) + P ({THT}) + P ({TTH}) + P ({TTT})

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

Now we have to consider only those outcomes which are favourable to the occurrence of the event F to answer the question "Suppose we are given that the first coin shows tail, what is the probability of occurrence of E?". That means we have to find out what **is common between E and F** or the condition given in F matches with that in the Event E. This can be found as given below.

$E \cap F = \{THH\}$

Now, the sample point of F which is favourable to event E is THH.

Thus, Probability of E considering F as the sample space = $\frac{1}{4}$

or Probability of E given that the event F has occurred = $\frac{1}{4}$

This probability of the event E is called the conditional probability of E given that F has already occurred, and is denoted by **P (E | F).**

Hence, $P(E \mid F) = \frac{1}{4}$

Note that the elements of F which favour the event E are the common elements of E and F, i.e. the sample points of $E \cap F$.

If E and F are the two events associated with the same sample space of a random experiment, the conditional probability of the event E given that F has occurred is given by:

$$P(E \mid F) = \frac{\text{Number of elementary events favourable to } E \cap F}{\text{Number of elementary events which are favourable to } F}$$

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}, \text{ provided } P(F) \neq 0$$

Example: A family has two children. What is the probability that both the children are boys given that at least one of them is a boy?

Solution :

Let E and F denote the following events :

E : both the children are boys

F : at least one of the children is a boy

Then E = {(*b*,*b*)} and F = {(*b*,*b*), (*g*,*b*), (*b*,*g*)}

Now E∩ F = {(*b*,*b*)}

Thus P(E I F) = $\frac{1}{3}$

We have used the formula :

P(E I F) = $\dfrac{\text{Number of elementary events favourable to } E \cap F}{\text{Number of elementary events which are favourable to F}}$

You can also verify the following formula in the above case.

P(E I F) = $\dfrac{P(E \cap F)}{P(F)}$ provided P (F) $\neq 0$

Check your progress-3

1. In a group of 100 sports car buyers, 40 bought alarm systems, 30 purchased bucket seats, and 20 purchased an alarm system and bucket seats. If a car buyer chosen at random, bought an alarm system, what is the probability they also bought bucket seats?

2. From the given data, find out the probability that a randomly selected person is male, given that he owns a pet?

	Have pets	Do not have pets	Total
Male	0.41	0.08	0.49
Female	0.45	0.06	0.51
Total	0.86	0.14	1

9.5 Theorem of Total Probability

Theorem of total probability "expresses the total probability of an outcome which can be realized via several distinct events".

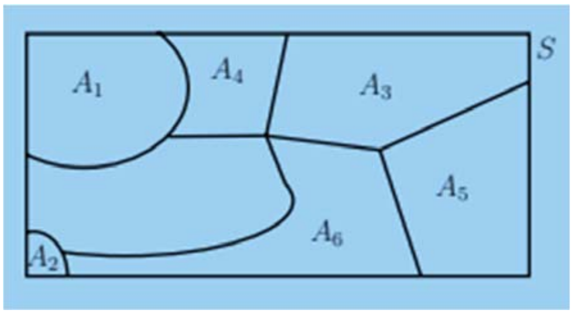
The collection of events $A_1, A_2, \ldots, A_n$ is said to be a **partition** in a sample space S if

(a) $A_1 \cup A_2 \cup \ldots \cup A_n = S$

(b) $A_i \cap A_j = \phi$ for all $i, j, i \neq j$

(c) $A_i \neq \phi$ for all i

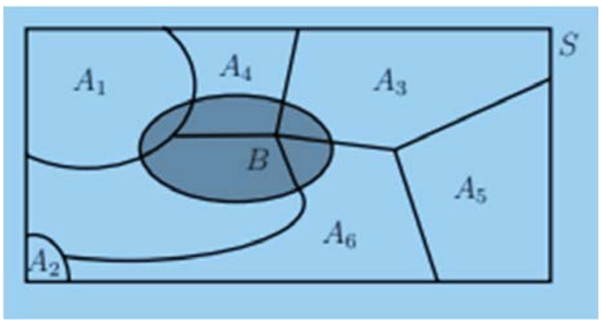
In essence, **a partition is a collection of non-empty, non-overlapping subsets of a sample space whose union is the sample space itself.**



If B is any event within S then we can express B as the union of subsets:

$B = (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_n)$

This is illustrated in the diagram that follows in which an event B in S is represented by the shaded region.



The bracketed events $(B \cap A_1), (B \cap A_2)$. . .$(B \cap A_n)$ are mutually exclusive (if one occurs then none of the others can occur). Using the addition law of probability for mutually exclusive events, we can write probability of event B as:

$P(B) = P(B \cap A_1) + P(B \cap A_2) + \cdots + P(B \cap A_n)$

Each of the probabilities on the right-hand side may be expressed in terms of conditional probabilities:

$P(B \cap A_i) = P(B \mid A_i)P(A_i)$ for all I $\left(\text{as } P(B|A) = \dfrac{P(B \cap A)}{P(A)}\right)$

Using these in the expression for P(B), above, gives:

$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + ... + P(B|A_n)P(A_n)$

$\sum_{i=1}^{n} P(B|A_i)P(A_i)$

Let us understand this theorem with an example. A person has undertaken a construction job. The probabilities are 0.65 that there will be strike, 0.80 that the construction job will be completed on time if there is no strike, and 0.32 that the construction job will be completed on time if there is a strike. Now we have to determine the probability that the construction job will be completed on time.

To solve this problem let us see if this comprise of two mutually exclusive events. We can see that there is either strike (B) or there is no strike (B'). The sum of these probabilities should be 1 . If we have strike then we cannot have 'no strike' and vice versa. That means events B and B' form partition of the sample space.

Hence we can write probability of strike as

$$P(B) = 0.65,$$

$$P(B') = P(\text{no strike}) = 1 - P(B) = 1 - 0.65 = 0.35$$

Now there is another event : Completion of work A . This can happen when there is strike and also when there is no strike for various reasons. This fits the requirement of application of theorem of total probability, we have

It is given that the probability of completion of work when there **is strike** (B) as

$P(A \mid B) = 0.32$

and probability of completion of work when there **is no strike** (B′) as

$P(A \mid B') = 0.80$

Now we can easily find out the probability of completion of work in any case as:

$P(A) = P(B)\,P(A \mid B) + P(B')\,P(A \mid B')$

$= 0.65 \times 0.32 + 0.35 \times 0.8$

$= 0.208 + 0.28 = 0.488$

Thus, the probability that the construction job will be completed in time is 0.488.

WATCH THIS VIDEO : https://www.youtube.com/watch?v=U3_783xznQI

Check your progress-4 :

Its given that 80% of people attend their family and doctor regularly; 35% of these people have no health problems cropping up during the following year. Out of the 20% of people who don't see their doctor regularly, only 5% have no health issues during the following year. What is the probability a person selected at random will have no health problems in the following year?

9.6 Bayes' theorem

Bayes' theorem, named after 18th-century British mathematician Thomas Bayes, is a mathematical formula for determining conditional probability. It provides a way to revise existing predictions or theories (update probabilities) given new or additional evidence. This theorem is the foundation of the field of Bayesian statistics.

Statement of the theorem:

If A_1, A_2, ... A_n are n non empty events which constitute a partition of sample space S

i.e., $A_i \cap A_j = \phi$ for all i, j $i \neq j$ which means

$A_1, A_2, \ldots A_n$ are pairwise disjoint

and $A_1 \cup A_2 \cup \ldots\ldots \cup A_n = S$

Also B is any event of non-zero probability, then

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)} \quad \text{for any } i = 1, 2, 3, \ldots, n$$

Proof: By formula of conditional probability, we know

$$P(A_i|B) = \frac{P(B \cap A_i)}{P(B)}$$

$$= \frac{P(A_i)P(B|A_i)}{P(B)} \qquad \text{(By multiplication rule)}$$

$$= \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)} \qquad \text{(By theorem of total probability)}$$

Remark: In Bayes' theorem the following terminology is applied.

The events $A_1, A_2, \ldots A_n$ are called hypotheses.

The probability $P(A_i)$ is called the priori probability of the hypotheses.

The conditional probability $P(A_i|B)$ is called a posteriori probability of the hypothesis A_i.

Let us try to understand this theorem with a practical example given below:

Illustration: Let us find out patients' probability of having liver disease if they are alcoholic. "Being an alcoholic" is the **test** for liver disease.

- **A** could mean the event "Patient has liver disease." Let us say 10% of patients entering a clinic have liver disease as per data available with the doctor. Probability of event A = P(A) = 0.10
- **B** could mean the confirmatory test that "Patient is an alcoholic." Let us assume that five percent of the clinic's patients are alcoholics. Probability of event B = P(B) = 0.05

You might also know that among those patients diagnosed with liver disease, 7% are alcoholics. This is **B I A:** the probability that a patient is alcoholic, given that they have liver disease, is 7%. Applying Bayes' theorem, we get

$$P(A|B) = (0.07 \times 0.1)/0.05 = 0.14$$

In other words, if the patient is an alcoholic, his/her chances of having liver disease is 0.14 (14%). This is a large increase from the 10% suggested by past data. But it's still unlikely that any particular patient has liver disease.

WATCH THESE VIDEOS NOW TO INTERNALIZE THE THEOREM.

Bayes' theorem : https://www.youtube.com/watch?v=HZGCoVF3YvM

Implications of Bayes' theorem :
https://www.youtube.com/watch?v=R13BD8qKeTg

Prior probability, in Bayesian statistical inference, is the probability of an event before new data is collected. This is the best way of assessing the probability of an outcome based on the current knowledge before an experiment is performed. **Posterior probability** is the revised probability of an event occurring after taking into consideration new information. Posterior probability is calculated by updating the prior probability by using Bayes' theorem. In statistical terms, the posterior probability is the probability of event A occurring given that event B has occurred.

Bayes' theorem thus gives the probability of an event based on new information that is, or may be related, to that event. The formula can also be used to see how the probability of an event occurring is affected by hypothetical new information, supposing the new information will turns out to be true.

Exercise on Bayes' theorem

1. The number of loans sanctioned by a particular branch of a bank under different heads and the percentage of defaults in each category is given above:

Types of Loan	Number of Loans Approved	Defaults (%)
Personal Loan	15	3%
Education Loan	5	1%
Housing Loan	10	2%
Car Loan	10	5%

If the loan application form picked at random for review is found to be of a person who has defaulted then find the probability that the application was for car loan.

2. A courier service company sends 30% of its orders by air, 50% by combination of bus and local transport and remaining 20% by train. Past record shows the courier is delivered late 2%, 7% and 5% of the time when orders are sent by air, bus local transport and train respectively. Find

   (i) the probability that the order will be delivered late

   (ii) the probability that the parcel delivered to a customer is sent by train if it is delivered late.

3. A young entrepreneur imports high tech machines for a startup venture. The imported machines are to be set up by an expert who is sent by the firm making these machines. From experience it is known that 80% of the times the expert is able to correctly set up the machines. If the setup is correctly done the machine produces 90% acceptable items and in case of an incorrect set up the machine produces only 50% acceptable item. If after a certain set up the machine produces an acceptable item followed by an unacceptable item find the probability that the machine is incorrectly set up.

4. An insurance company insures scooter drivers, car drivers and bus drivers in the ratio 4:5:3. The probability of a scooter driver, car driver and bus driver meeting with an accident is 0.7%, 0.4% and 1.2% respectively. If an insured

person meets with an accident find the probability that the person is a scooter driver.

5. Two cards from a pack of 52 cards are lost. From the remaining cards of the pack a card is drawn at random and is found to be spade. Find the probability that the lost cards are both spades.

6. A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested. If 0.2% of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive?

Check your progress - 5

1. Car Repair Diagnosis

   The manager of a car repair workshop knows from past experience that when a call is received from a person who is stuck up far away and has problem starting the car, the probabilities of various troubles are as follows:

Event	**Trouble**	**Probability**
A_1	Battery problem	0.4
A_2	No petrol	0.3
A_3	Flooded	0.1
A_4	Some other reason	0.2

   Assuming that no two faults occur simultaneously and E be the event that the car starts if instructions given by the manager are followed by the driver with probabilities:

   $P(E|A_1) = 0.3$, $P(E|A_2) = 0$, $P(E|A_3) = 0.8$, $P(E|A_4) = 0.5$

   (a) If a person follows the instructions given by the manager, what is the probability that the car starts?

   (b) If the car starts on following the instructions of the manager, find the probability that car had a battery problem.

2. In a factory which manufactures bulbs, units A, B and C manufacture respectively 25%, 35% and 40% of the bulbs. Of their outputs, 5, 4 and 2 percent are respectively defective bulbs. A bulbs is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the unit B?

Summary

1. A random experiment is a process by which we observe something uncertain
2. An outcome is a result of a random experiment.
3. The set of all possible outcomes is called the sample space. The sample space is defined based on how you define your random experiment.
4. A trial is a particular performance of a random experiment.
5. An event is a subset of the sample space.
6. The ratio of number of favourable outcomes to the total number of outcomes is defined as the probability of occurrence of an event.
7. We have Independent or dependent Events, Impossible and Sure Events, Exhaustive Events, Mutually Exclusive Events according to the probabilities with reference to the sample space.
8. If E and F are two events associated with the same sample space of a random experiment, the conditional probability of the event E given that F has occurred,

$$P(E\,|\,F) = \frac{\text{Number of elementary events favourable to } E \cap F}{\text{Number of elementary events which are favourable to } F}$$

$$P(E\,|\,F) = \frac{P(E \cap F)}{P(F)}, \text{ provided } P(F) \neq 0$$

9. Theorem of total probability "expresses the total probability of an outcome which can be realized via several distinct events".

if

(a) $A_1 \cup A_2 \cup \ldots \cup A_n = S$

(b) $A_i \cap A_j = \phi$ for all i, j

(c) $A_i \neq \phi$ for all i

Then :

$P(B) = P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + \cdots + P(B \mid A_n)P(A_n)$

$$= \sum_{i=1}^{n} P(B|A_1)P(A_i)$$

10. Bayes' Theorem If $A_1, A_2, \ldots$ An are n non empty events which constitute a partition of sample space S. Also B is any event of non zero probability, then

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)} \text{ for any } i = 1,2,3, \ldots,n$$

Practical and project work

1. Find out different examples of practical applications of Total Probability theorem and Bayes' theorem in finance, research and industry

2. Search for data and frame a problem to be solved by Bayes' theorem. Use excel sheet to calculate the result.

Further exploration

1. https://www.cimt.org.uk/projects/mepres/alevel/stats_ch1.pdf

Solutions to Check your Progress - 1

1. Observing the Number of people voted in a constituency.

2. S= {DDD, DDN, DND, NDD, DNN, NDN, NND, NNN}

Solutions to check your Progress-2

1. Independent events

   A : Person having black hair

   B : Person working in a MNC

   Dependent events : Traffic and Condition of road Accident

2. Impossible event : Sun rising in the west

   Sure event : Getting sum of number ≤ 12 when a pair of dice are rolled

3. Exhaustive events

   A: Getting a head in a single throw of a coin

   B: Getting a tail in a single throw of a coin

4. Mutually exclusive events

   A: A person running forward

   B: A person running backward avB=0

5. Mutually exclusive and exhaustive events

   A: getting an even number in a throw of a die

   B: getting an odd number in a single throw of a die.

Solution to check your Progress - 3

1. Let A denotes the event of car buyer buying alarm system and B denotes the event of car buyer buying bucket seat

P(A) = 40%, or 0.4.

P(A∩B) = $\frac{20}{100} = 0.2$

Substitude the values in the formula

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.2}{0.4} = 0.5$$

The probability that a buyer bought bucket seats, given that they purchased an alarm system, is 50%.

2. Let M stands for event of person being male and P stands for a event that person is a pet owner, we have:

$P(M \cap P) = 0.41.$

$P(P) = 0.86$

Substituting in the formula:

$$P(M \mid P) = \frac{P(M \cap P)}{P(M)} = \frac{0.41}{0.86} = 0.477, \text{or } 47.7\%$$

Solution to check your Progress – 4

1. Let D denotes the event of seeing the doctor regularly and H denotes the event of facing no health issues the following year.

   Probability of those who see doctor regularly, $P(D) = 0.80$

   Probability of those who do not see the doctor regularly, $P(D') = 0.20$

   Probability of those who see the doctor given that have no health issues $P(H \mid D) = 0.35$

   Probability of those who do not see doctor and have no health issues $P(H \mid D') = 0.05$

   $P(H) = P(D)\,P(H \mid D) + P(D')\,P(H \mid D')$

   $= 0.80 \times 0.35 + 0.20 \times 0.05 = 0.28 + 0.01$

Answer to exercise on Baye's Theorem

1. 5/12
2. 51/1000, 10/51
3. 25/61
4. 1/3
5. .051
6. 198/697

Solution to Check your Progress - 5

1 (a) We need to compute P(E)

Using theorem of total probability, we get

P(E) $= P(A_1)\,P(E \mid A_1) + P(A_2)\,P(E \mid A_2) + P(A_3)\,P(E \mid A_3) + P(A_4)\,P(E \mid A_4)$

$= 0.4 \times 0.3 + 0.3 \times 0 + 0.1 \times 0.8 + 0.2 \times 0.5$

$= 0.12 + 0 + 0.08 + 0.10$

$= 0.3 \quad \dots (1)$

The probability that the car starts is 0.3

(b) We use Bayes' formula to compute the posteriori probability $P(A_1 \mid E)$:

$$P(A_1 / E) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{4} P(A_j)P(B|A_i)}$$

$$= \frac{0.4 \times 0.3}{0.3} \qquad \text{(Using 1)}$$

$= 0.4$

Thus the probability that the car had battery problem if it started on following managers instruction is 0.4.

2. Let B_1, B_2, B_3 be the event that the bulb is manufactured by units A, B and C respectively and E be the event that bulb manufactured is defective

$P(B_1) = 0.25$, $P(B_2) = 0.35$ and $P(B_3) = 0.40$

$P(E \mid B_1)$ = Probability that the bolt drawn is defective given that it is manufactured by unit A = 5% = 0.05

Also $P(E \mid B_2) = 0.04$, $P(E \mid B_3) = 0.02$.

$$P(B_2 \mid E) = \frac{P(B_2) \times P(E \mid B_2)}{P(B_1) \times P(E \mid B_1) + P(B_2) \times P(E \mid B_2) + P(B_3) \times P(E \mid B_3)}$$

$$= \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02}$$

$$= \frac{0.014}{0.0345} = \frac{28}{69}$$

Chapter X

Descriptive Statistics

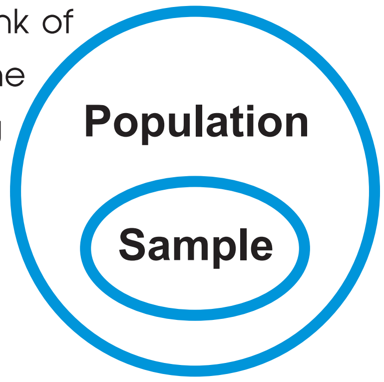
Learning Objectives:

After completion of this unit the students will be able to

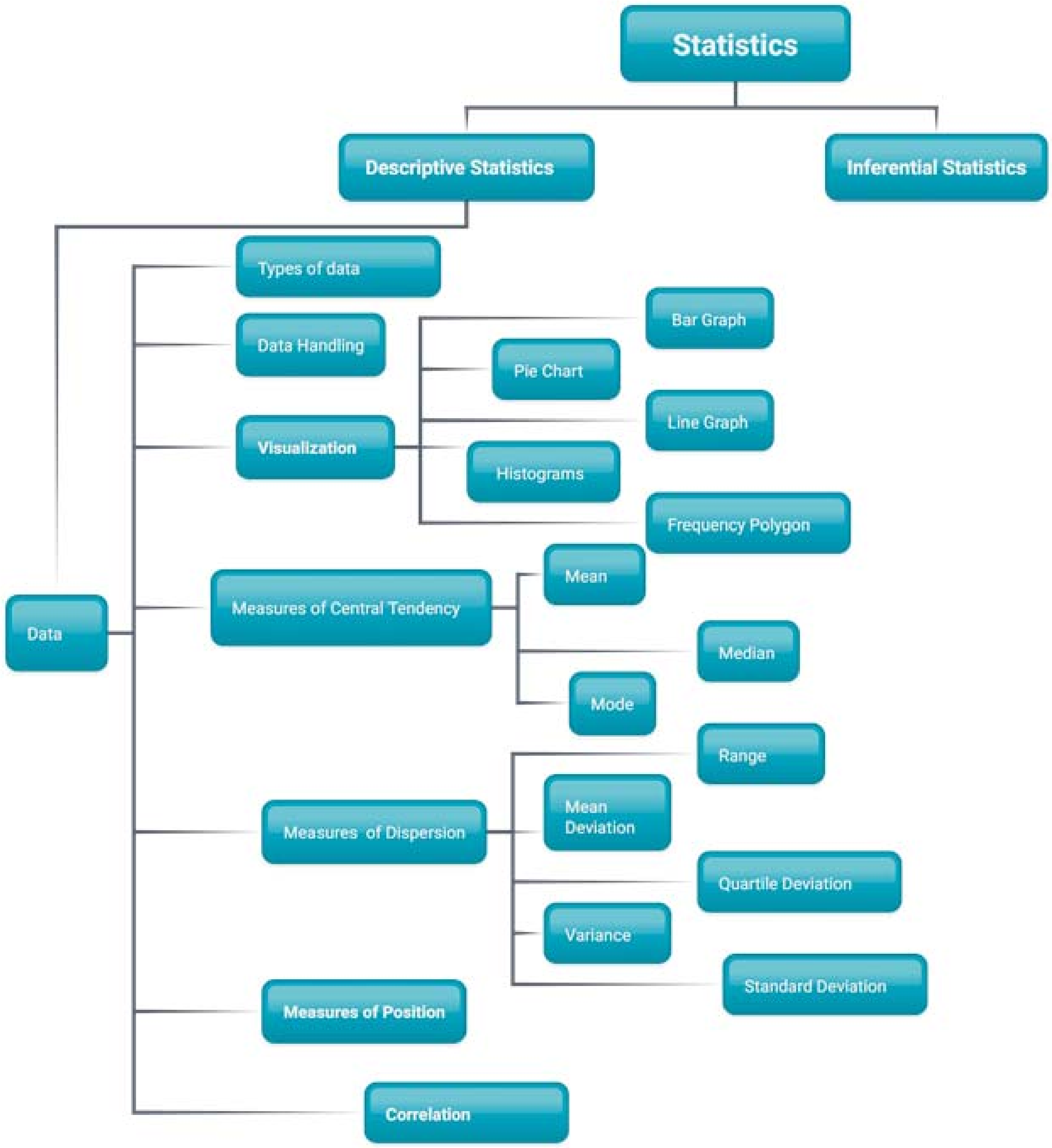
- develop an understanding of everyday data
- understand the organization, visualization and analysis of data
- create and analyze the graphical displays to summarize data
- draw meaningful conclusions from data
- make comparison among two distributions
- translate real world problems and make meaningful inferences out of it

Before you start you should know about "The population and samples."

In Statistics, population is not the same as we usually think of people in our village, town, state or country. It includes all the members of a defined group that we deal for making decisions whereas a sample is the subset of population.



Concept Map



Activity for recap

Activity 1: Suppose your class of 25 students had appeared for a test and marks obtained were;

0, 1, 1, 1, 2, 2, 3, 3, 3, 3, 4, 5, 5, 5, 6, 6, 7, 7, 7, 7, 8, 8, 9, 10, 10

- How can you display and organize this given set of data?
- What conclusions can teacher draw from this data?
- What is the average of the data?

10.1 Introduction

In earlier classes, you have studied some basic concepts of statistics so you may be familiar with this through various kinds of data in your day to day life like newspaper, books, surveys, observations, etc. It plays an important role in our life for understanding the up-down in economy, population growth, climate change, weather prediction, election results and many more. We use statistics without realizing it. Every day you speak so many statistical statements unknowingly.

The facts or numerals collected for a specific purpose is called data. The word data is plural so a data set is always a group of many numbers.In fact data speaks or we can say that data can be transformed into useful information. Let us have a look on some of the newspaper headlines

(i) India to become a 5 trillion dollar economy by 2025

(ii) India's population to surpass that of China around 2024: UN

(iii) States' tax share to stay at 42% : XV Finance Panel

(iv) Inflation may drop to 2.4% in financial year 2021: RBI

We can find some sort of data in the above mentioned headlines. These data are based upon some or other information that can be verified using appropriate mathematical tools. Hence the science of collecting, studying and analyzing numerical data is known as statistics.

This is basically categorized into two branches. First, Descriptive Statistics deals with the collecting, summarizing and interpreting the data so that it can be easily understood. Second, Inferential Statistics deals with the methods for obtaining and analyzing data to make inferences by performing estimations and hypothesis tests. In this class we shall study only descriptive statistics; the other part of inferential statistics will be taken up in class XII.

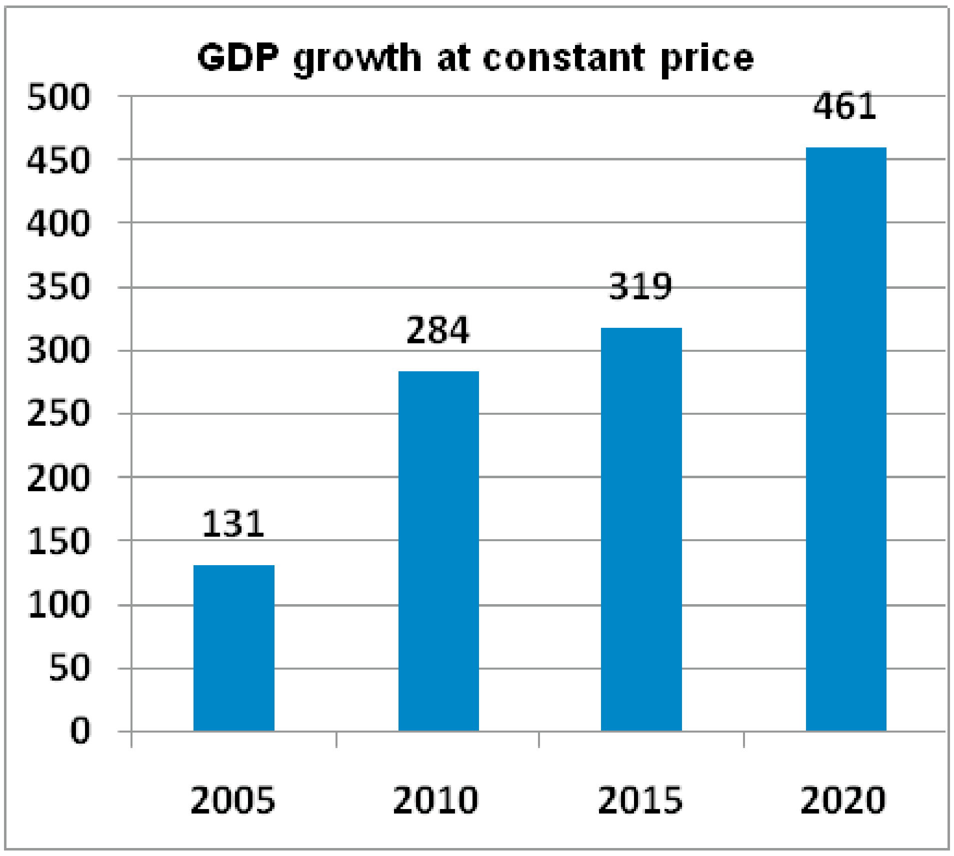
Note

- The word data is plural so a data set is always a group of many numbers. One piece of information is called a "datum"
- Statistics is concerned with the
  - (i) Collecting data
  - (ii) Organizing data
  - (iii) Summarizing data
  - (iv) Analyzing data
  - (v) Drawing conclusions / making inferences from data
- Descriptive statistics are just descriptive. They do not include generalization beyond the data.

Historical Note

Descriptive Statistics is considered to be originated during the census taken by the Babylonians and Egyptians (4500-3000 BC). They used to maintain the record for number of livestock each person owned and the crops each citizen harvested yearly.

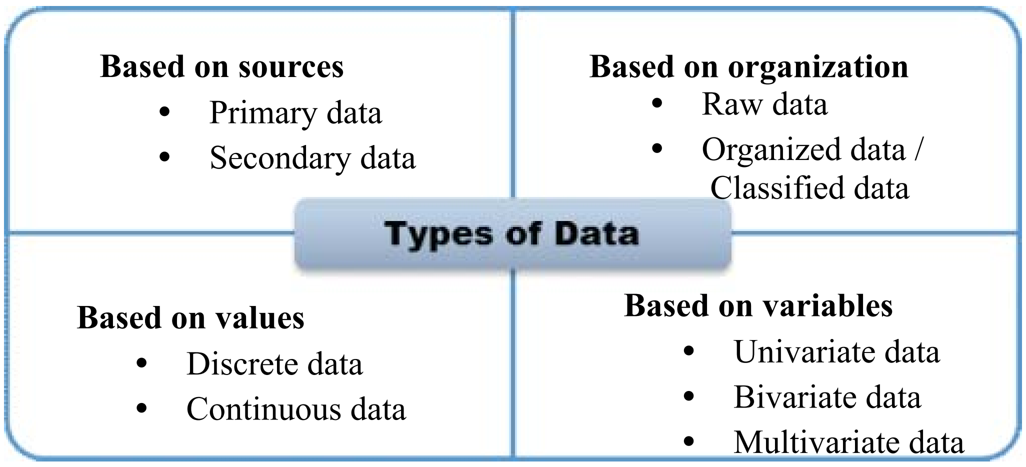
The figure given on next page gives India's Foreign Exchange Reserves (in US billion dollar) in recent years, and it make us ponder how the numbers are increasing regularly. Explanation for this rise may be anything but these descriptive statistical data offer an insight.



Source: Weekly statistical supplement of RBI

https://m.rbi.org.in/Scripts/WSSViewDetail.aspx?TYPE=Section&PARAM1=2

10.2 Types of data



Data based on sources (Primary / Secondary data)

Based upon the sources of collection, data can be either Primary or Secondary. If we collect it directly from the field or from individual (surveys, observations, case studies), the data is called Primary Data. If it is taken from already collected data then such data is called Secondary Data.

Example 1: The teacher collected the information about the height of students in her class.

If teacher collects the data by measuring height of each student then data so collected is called the Primary Data, whereas, if teacher refer medical records of the students to note the height of students is called Secondary Data.

Data based on organization (**Raw / organized data**)

When we first collect the data, it is mostly unorganized and it is very difficult to make any meaningful observations on it. This first hand collected data is called Raw Data.

When we organize the raw data in order to draw any meaningful conclusion out of them, it is called organized/classified data.

Raw data: Raw data is unclassified or highly disorganized data. They require suitable organization in order to draw any meaningful conclusion out of them. For example, ODI run secured by Virat Kohli in the year 2019 given in below table is raw data.

3	104	46	45	43	60	44	116	123	7	20	18	82
77	67	72	66	26	34	1	120	114	4	0	85	23

Source: http://www.howstat.com/cricket/statistics/players/PlayerProgress Bat_ODI.asp?PlayerId=3600&Year=2019

This data needs to be organized in a proper order before any systematic statistical analysis is undertaken.

Note: Raw data is also known by primary data/first hand data.

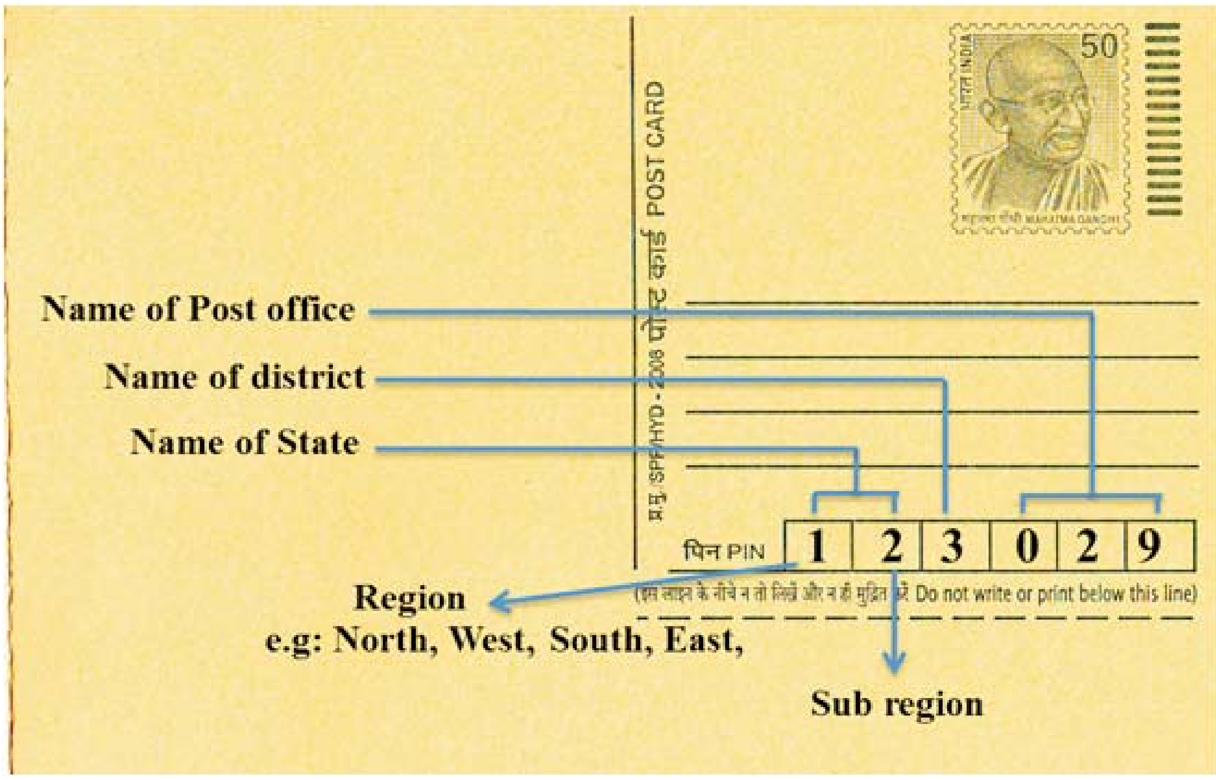
Question: Do you agree that classified data is better than raw data? Explain with an example from your daily life.

Activity 2

- Collect data of your daily pocket expenditure for a week/month and put it in a table. Find the number of observations and arrange the daily data.

Activity 3

PIN (Postal Index Number) code is the post office numbering code system used by the postal service of India. It is a 6 digits long code with each of the digits denoting a particular meaning. Here is how your PIN code is decided.



Explore the concept, how categorizing the areas with Pin codes helps postman to sort the letters and deliver at exact place.

Data based on variables (Univariate, Bivariate, Multivariate data)

(I) Univariate data involves only one variable (as Uni means one), for example, height of all the students in your class. Thus, their analysis (charts, averages) is the simplest form of analysis since information deals with only one particular quantity that changes.

Example 2: Temperature of Delhi during last seven days(in degree Celsius)

Temperature (^{0}C)	40	41	37	38	34	38	36	39	40

Here only temperature is a variable, thus this is called univariate data.

(ii) **Bivariate data** involves two different variables. The analysis is done to find out the connection between these two variables mainly the causes and relationships.

Example 3: Demand curve correctly explains the bivariate data where quantity demanded of a good has an inverse relationship with the price (keeping other factors constant like income).

Example 4: Following bivariate data depicts age and average height of a group of babies and kids.

Age (**in months**)	3	6	9	12	24	36	48	60
Height (**in cms**)	58	64	68	74	81	89	95	102

other example for bivariate data can be distance time graph, inflation, unemployment graph, caloric intake versus weight graph which compares two variables.

(iii) **Multi-variate data** involves more than two variables where analysis is done to understand the interactions between different fields in the data

Example 5: Your grade in class XI depends on the performance in various subjects (i.e multiple variables) you have opted.

Example 6: Profit of a company depends on the various statistics like cost of raw materials, labor, marketing expenditures, sales etc.

Univariate data	**Bivariate data**	**Multi-variate data**
● involves a single variable ● does not deal with causes or relationships ● the major purpose of univariate data analysis is to describe ❖ frequency distributions ❖ bar graph, histogram, pie chart, line graph ❖ central tendency (mean, mode, median) ❖ dispersion (range, quartiles, variance, standard deviation)	● involves two variables ● deals with causes or relationships ● the major purpose of bivariatedata analysis is to describe ❖ analysis of two variables ❖ correlations ❖ comparisons, ❖ relationships, causes, ❖ explanations	● involves more than two variables ● deals with explanatory purpose ● There exist various ways to analyze the multivariate data depending on the goals.

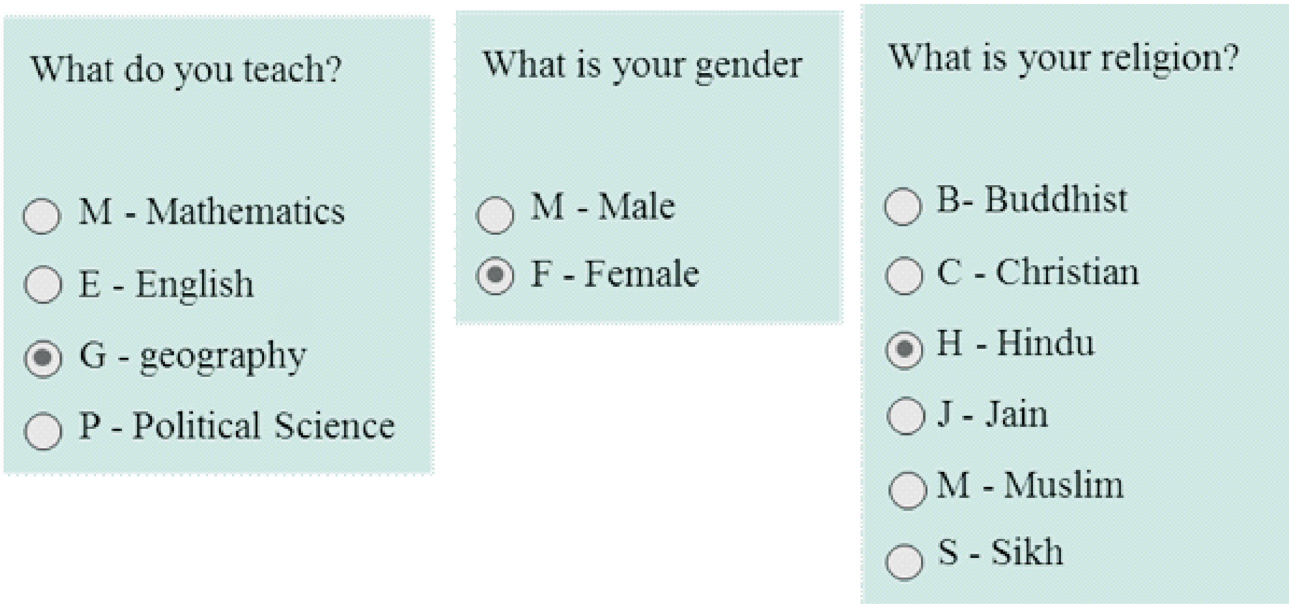
10.3 Characterizing the data based on variables / Data on various scales

We collect data in different forms; sometimes we collect data in categories, sometimes in positional value and sometimes in absolute value. More precisely, we collect data on different scales of measurement. Classification of data based upon the method of categorizing, counting and measuring uses four common types of measurement scales- nominal, ordinal, interval, and ratio.

10.3.1 Nominal level measurement:

Nominal level measurement differentiates data based upon their names or type of category they belong to.

Example 7: A sample of teachers in your school categorized according to subject they teache (Mathematics, English, Geography, Political Science, etc.) is an example of nominal level measurement. Putting them as per their gender (male, female), marital status (married/unmarried), religion, is also an example of this kind of measurement.



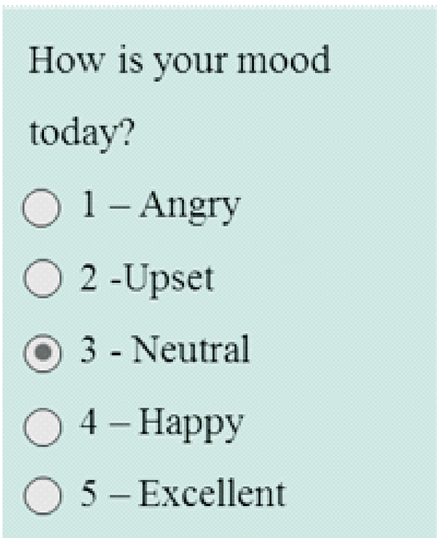
- The categories don't have to be numerical.
- It put the data into mutually exclusive (non-overlapping), exhausting categories where ranking or no order can be applied to the data.
- Each observation belongs to one of several distinct categories.
- They can be called labels (as we simply labeling the variables)

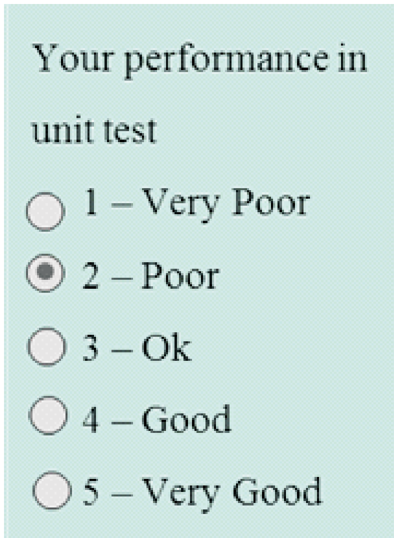
10.3.2 Ordinal level Measurement:

Ordinal level measurement allows categorizing the data and these categories can be ordered, or ranked but precise measurement of differences does not exist.

Example 8: You can put Indian cricket team players as superior, average, or poor batsman. But precise measurement can not be done as sometime they may perform differently.

Therefore ordinal level of measurement classifies data into categories that can be ranked; however, precise differences between the ranks do not exist.





- Observations can be placed into categories that can be ranked.
- The interval between each value of data in scale is not always same.

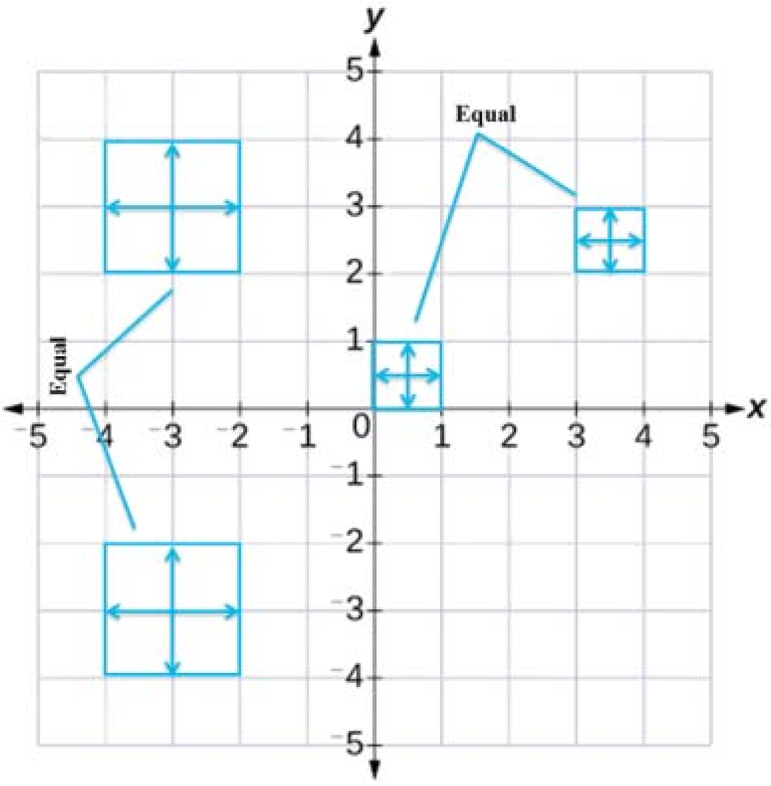
10.3.3 Interval Scale measurement

Interval level measurement specifies the equal distance between each interval, along with categorizing and putting into orders.

Example 9: 100 rupee to 200 rupee is the same interval as 700 rupee to 800 rupee.

Locations in Cartesian coordinate system is another perfect example, where

- Difference between consecutive values is always the same.
- Interval can represent below zero values as well.



10.3.4 Ratio Scale measurement

In ratio level measurement the observations have a value of zero as well along with the having equal intervals i.e Ratio level of measurement has all the features of interval measurement, in addition to existence of a meaningful true zero value.

Examples include height, weight, area, length, time duration etc.

Ratios are always more meaningful as they depicts 'how much' or 'how many' of something.For example, when we say that

i. a car takes double time than train to travel the same distance.

ii. a person has twice the height of his son.

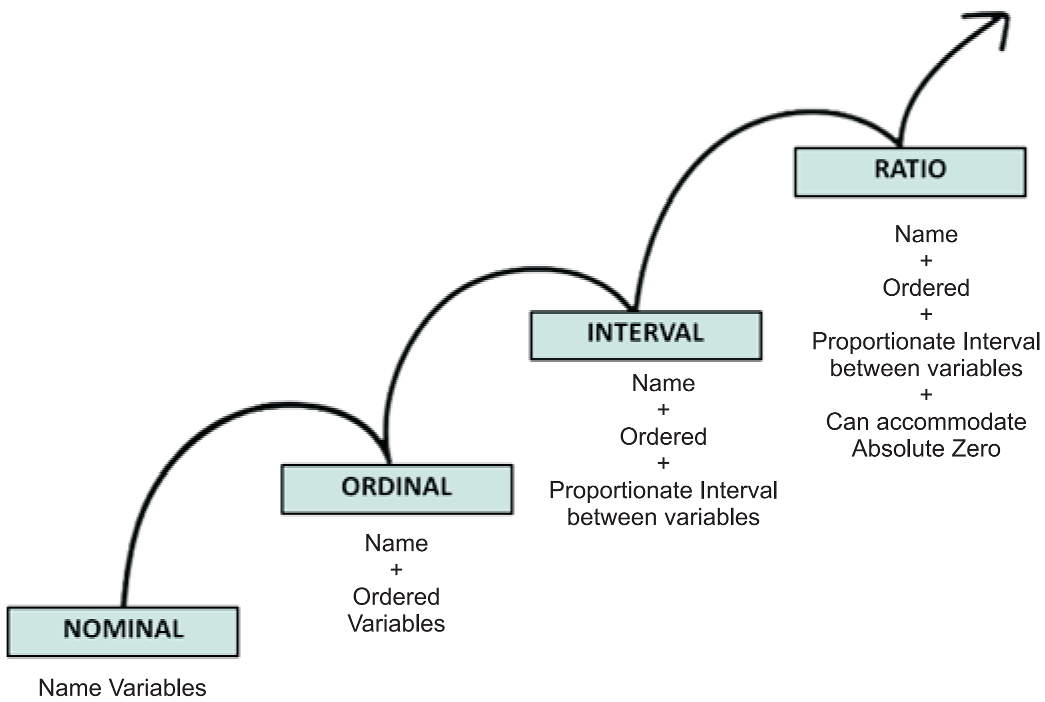
Then ratio between them is 2 to 1 (true ratios).

True ratios are considered to exist when the same variable is measured on two different members of the population.

Note

The main difference between interval and ratio scales comes from their ability to dip below zero. Interval can represent below zero values. For example, you can assign a point below 0 in Cartesian coordinate or temperature below 0 degree can be measured.

But ratio variables never fall below zero value. For example, height and weight is always measured to be 0 and above.



Classification	Graphical measures	Measures of central tendency	Measures of dispersion
Nominal	Bar graphs Pie charts	Mode	Binomial or multinomial variance
Ordinal	Bar graphs Histogram	Median	Range
Interval	Histogram areas are measurable	Mean	Standard deviation
Ratio	Histogram areas are measurable	Geometric Mean, Harmonic Mean	Coefficient of variation

Table adapted from Afifi, A., S. May, and V.A. Clark. 2012. Practical multivariate analysis 5th edition. CRC Press, Taylor and Francis Group, Boca Raton, FL.

10.4 Data representation and visualization

A picture is worth a thousand words. The phrase correctly depicts the essence of presenting data in the form of diagrams and graphs (charts). There are a large number of diagrams being used for presentation of data. The selection of particular diagram depends on the nature of data, objective of presentation and the ability and experience of the person doing this task. Some popular types of diagrams are discussed below;

i. Bar Graph

ii. Pie Chart

iii. Line Graph

iv. Histograms

v. Frequency Polygons

Graph helps in showing the relation between dependent and independent variables.

Example 10 : you might have observed that amount spent on petrol is directly dependent on the distance travelled by the vehicle. Here one quantity affects the another. So here quantity of petrol consumed is an independent variable and amount of petrol bill is the dependent variable. These kind of relationship between such variables can be shown through a graph.

10.4.1 Bar Graph

Bar graph is pictorial representation of data having rectangular bars of equal width. These bars are placed on equal space on one of the axis (either x or y) i.e bars can be either vertical or horizontal. Height of the bars depends on the given data where lower end of the bar touches the base line such that the height of each bar starts from zero units.

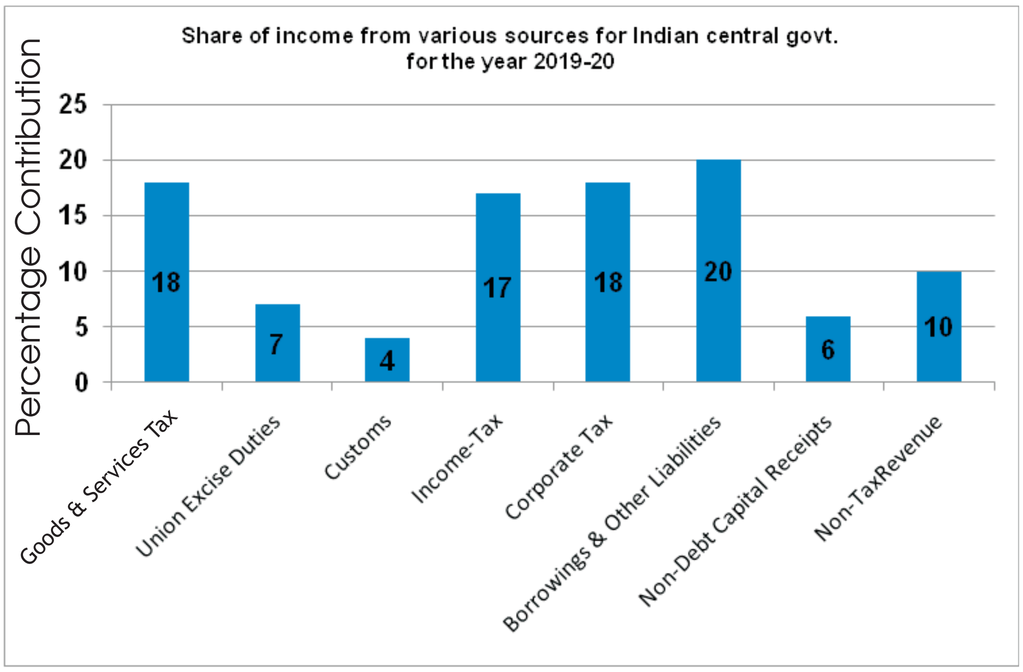
- Every bar depicts only one characteristics of the data.
- The distance between the bars should be equal.
- These bars can be either vertical or horizontal.

- Bars of a bar diagram can be visually compared by their relation height and accordingly data can be comprehended quickly.

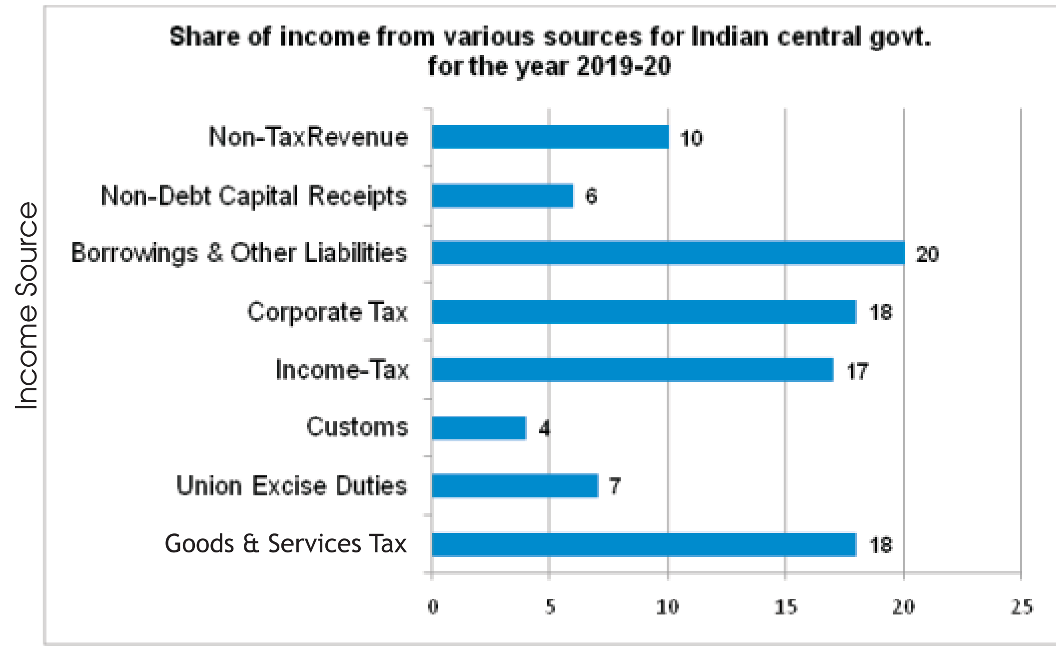
Example 11: Let us take data disclosed in union budget 2020-21 for percentage shares of earning for Indian central govt. for the year 2019-20

Income source	**Percentage contribution**
Goods & Services Tax	18%
Union Excise Duties	7%
Customs	4%
Income-Tax	17%
Corporate Tax	18%
Borrowings & Other Liabilities	20%
Non-Debt Capital Receipts	6%
Non-Tax Revenue	10%

(Source: https://www.indiabudget.gov.in/)



Income Source



Percentage Contribution

By observing the above bar graphs answer the following questions:

(i) What is the maximum earning source of government?

(ii) Which two sources credited equal amount of money to government?

10.4.2 Pie Chart

Pie Chart is a circular chart in which we divide the circle into multiple sectors (pie slices). It is used when comparison of a component part is required with other component and the total.

- Pie diagrams is also known as "Angular Circle Diagram"
- To construct it, we use the fact that total of all given values corresponds to the total number of degrees in the circular arc i.e 360^0
- Each sector represents a particular component as a part of the whole
- It is used to show relative sizes

Steps of Pie Diagram:

(i) Construct a circle and divide it into number of sectors to represent the components.

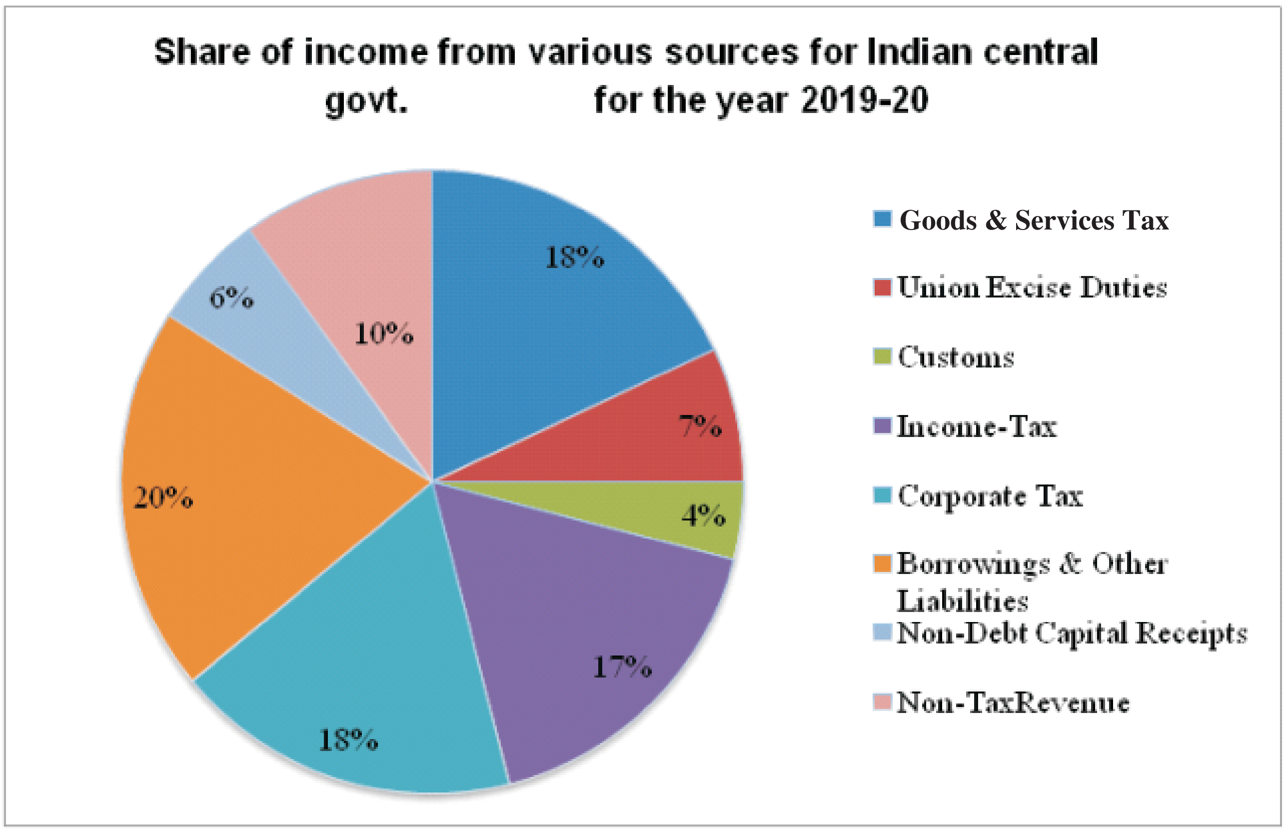
(ii) Convert the value of various components into percentage value of total.

(iii) The percentage values are converted in to corresponding degrees in the circle. Since one circle contains 360^0 and percentage value of all the items are equal to 100, therefore our percentage value is represented by $(360^0)/100 = 3.6^0$ to get size of angles.

(iv) Take base line to draw the angle represented by first component. The new line will become the base for second components angular representation. Repeat these procedures till all the components are represented.

Example 12: Data given in previous example can be represented through Pie Chart as follows:

Income source	**Percentage contribution**	**Share as a component of 360^0**
Goods & Services Tax	18%	$\frac{18}{100} \times 360^0$
Union Excise Duties	7%	$\frac{7}{100} \times 360^0$
Customs	4%	$\frac{4}{100} \times 360^0$
Income - Tax	17%	$\frac{17}{100} \times 360^0$
Corporate Tax	18%	$\frac{18}{100} \times 360^0$
Borrowings & Other Liabilities	20%	$\frac{20}{100} \times 360^0$
Non - Debt Capital Receipts	6%	$\frac{6}{100} \times 360^0$
Non - Tax Revenue	10%	$\frac{10}{100} \times 360^0$

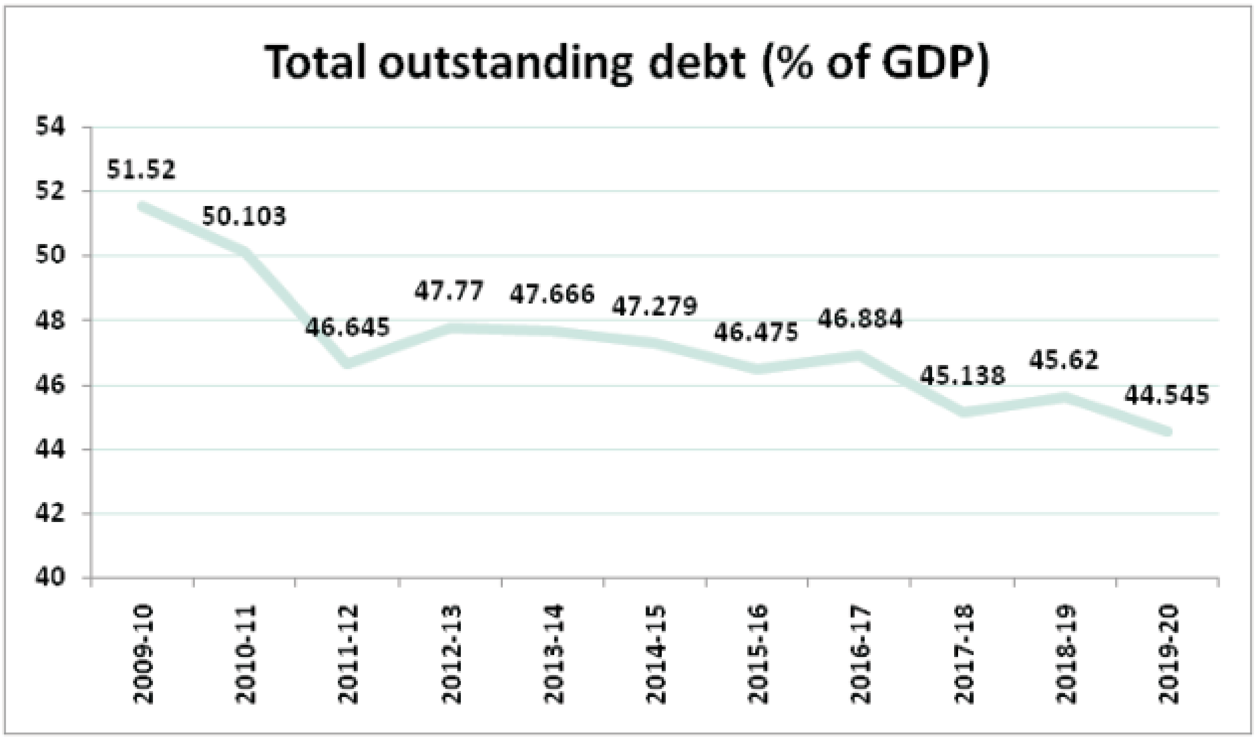


10.4.3 Line Graph

Line graph connects a series of successive data points and shows how one variable behaves over the other variable by representing each value as a dot and connecting the dots to form a line. Usually it is used to relate the time variable with other variables.

- Each dot depicts a different data point
- More than one line may be plotted in the same axis for comparison purpose.

Example 13: Following line graph presents the total outstanding debt (% of GDP) of India, which is the accumulation of borrowings over the years. A higher debt implies that the government has a higher loan repayment obligation over the years.



Source: Economic Surveys 2003-04 to 2018-19 and
https://m.rbi.org.in/Scripts/BS_ViewBulletin.aspx?Id=18703

Fact check : Total outstanding debt of the government has decreased from 55.5% of GDP in 2000-01 to 50.1% of GDP in 2020-21 (estimate). The FRBM Act sets a target of 40% of GDP for outstanding debt to be met by 2024-25.

10.4.4 Histograms

Histogram is like bar graph but does not have gaps between the bars. It is used for continuous class intervals (never drawn for a discrete variable) where class frequencies are represented by area of the corresponding bars.

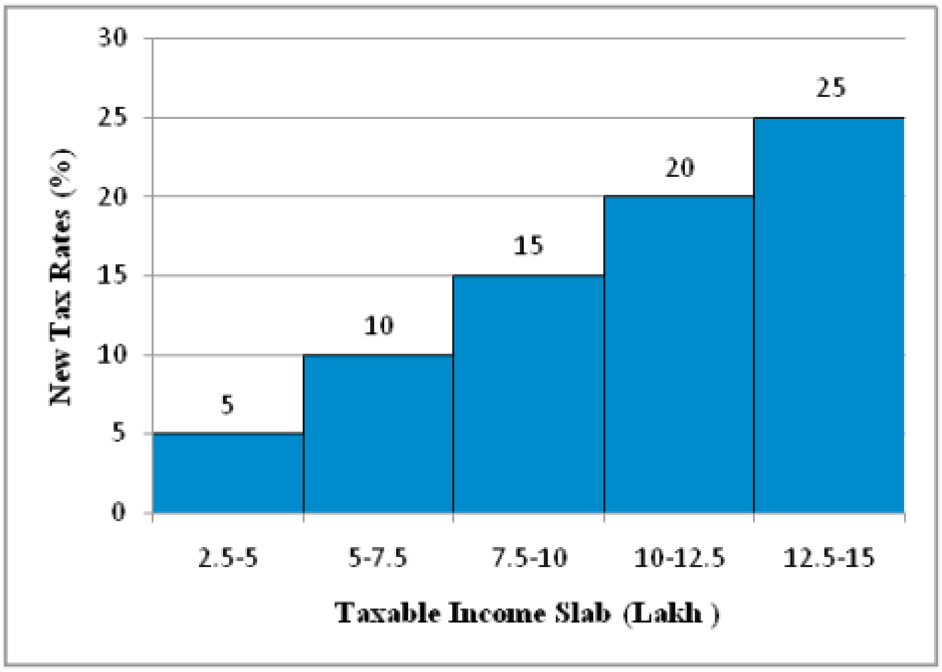
- It is a graph with consecutive bars.
- Height of every bar is equal to the corresponding frequency of the class intervals, taller the bars means more data falls in that particular range.

Example 14: Consider the frequency distribution representing the new tax rates in country as proposed in India's union budget 2020-21

Taxable Income Slab (Lakhs)	New Tax Rates (%)
2.5-5	5
5-7.5	10
7.5-10	15
10-12.5	20
12.5-15	25

Source: PRS India

https://www.prsindia.org/parliamenttrack/budgets/union-budget-2020-21-analysis



In case of **unequal class intervals** we need to first adjust the frequencies before constructing the histograms, as width of rectangular bars is going to be unequal which doesn't suit the definition of histogram.

Steps to adjust frequencies:

Step1: Determine minimum class size. In below example, the minimum class size is 5

Step 2: The lengths of the rectangles (frequencies) are then modified to be proportionate to the minimum class size.

- Adjusted frequency of a class = (Minimum class size)/(Class size) x Frequency

- By doing so we will see that frequencies of the classes with minimum class size are not changed and frequencies of all other classes are adjusted.

Example 15 : Prepare the histogram for following data

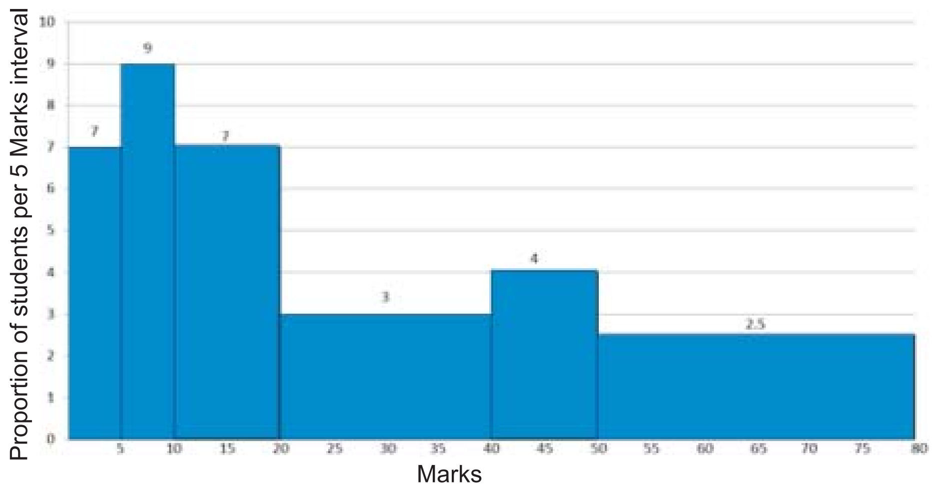
Marks	0-5	5-10	10-20	20-40	40-50	50-80
No of students	7	9	14	12	8	15

Here class intervals are unequal so we need to adjust the frequencies for preparing the histogram.

The histogram for adjusted frequencies is shown as under

Marks	**Frequency (No of students)**	**Class size**	**Adjusted frequency**
0-5	**7**	**5**	$\frac{5}{5}\times 7 = 7$
5-10	**9**	**5**	$\frac{5}{5}\times 9 = 9$
10-20	**14**	**10**	$\frac{5}{10}\times 14 = 7$
20-40	**12**	**20**	$\frac{5}{20}\times 12 = 3$
40-50	**8**	**10**	$\frac{5}{10}\times 8 = 4$
50-80	**15**	**30**	$\frac{5}{30}\times 15 = 2.5$

Now we have calculated the class sizes of 5 marks in each case, so the correct representation of histogram with unequal class intervals is constructed in below figure



Question 1:

Discuss the difference between bar graph and histogram from charts given in examples of respective concepts.

10.4.5 Frequency Polygon

A frequency polygon is similar to line graph but it is used for a continuous frequency distribution. The intervals in the continuous distribution are represented by the midpoint of each corresponding interval and mid points and corresponding frequencies are plotted as points in XY plane. All points are joined using free hand. When joined by line segments we obtain a figure and to complete the polygon we assume a class interval with frequency zero.

Frequency polygon can also be drawn independently without drawing histograms. For this, we need to directly find out the mid-points of respective class-intervals. These mid points are called as Class marks and calculated by

$$\text{Class Mark} = \frac{\text{Upper Limit + Lower Limt}}{2}$$

A. Creating frequency polygon without histogram:

Step 1: Mark the values of variable on Horizontal x-axis and frequencies on vertical y-axis.

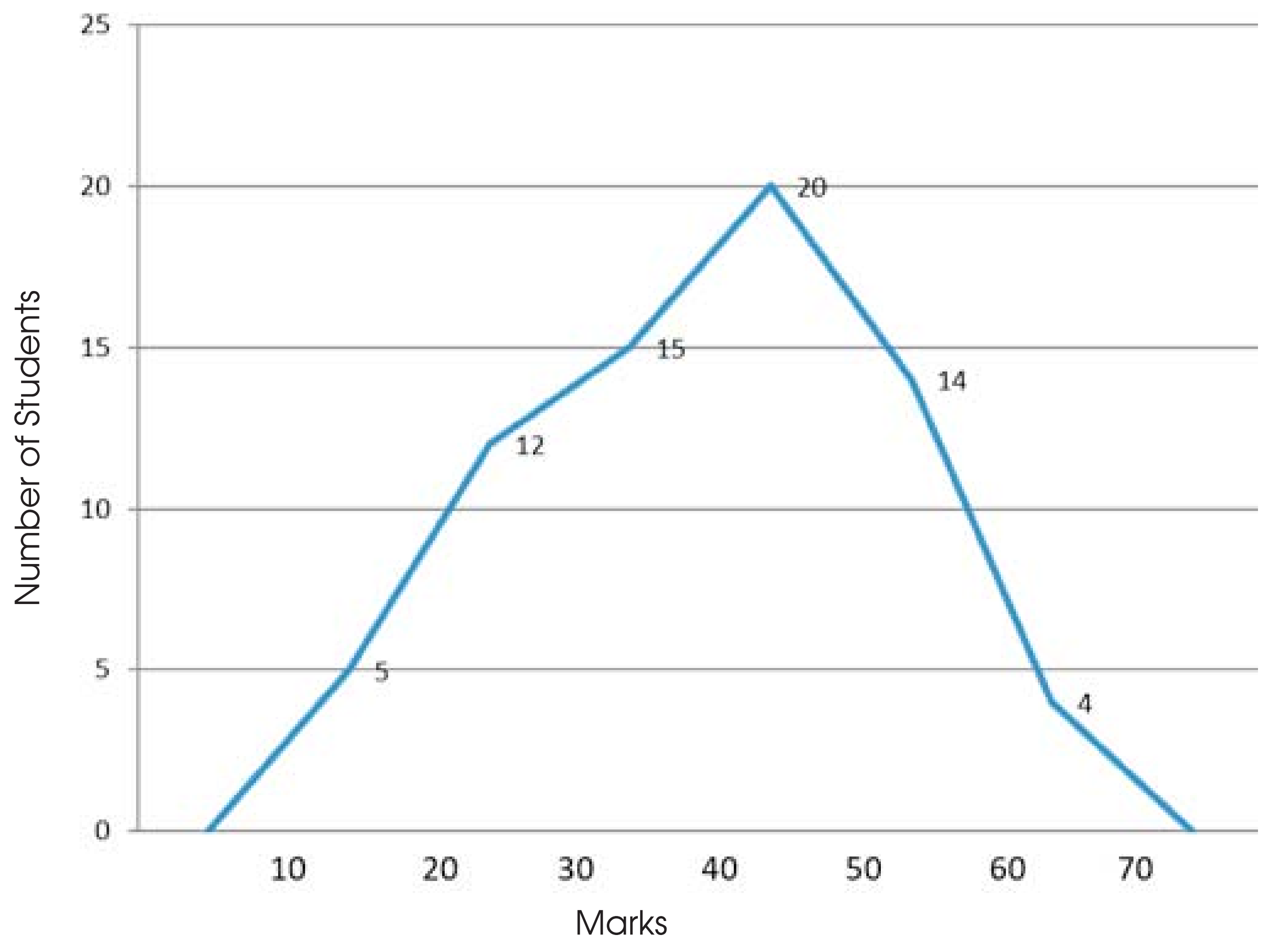
Step 2: Connect the points so drawn by a straight line.

Step 3: Connect the extreme points to the base assuming to have zero frequency.

Example 16: Consider the marks, out of 100, obtained by 70 students of a class in a test, given in table below:

Class Interval	10-20	20-30	30-40	40-50	50-60	60-70
Frequency	5	12	15	20	14	4

Draw a frequency polygon corresponding to this frequency distribution table.



B. Creating frequency polygon with histogram:

Step 1: Draw histogram of the given frequency distribution

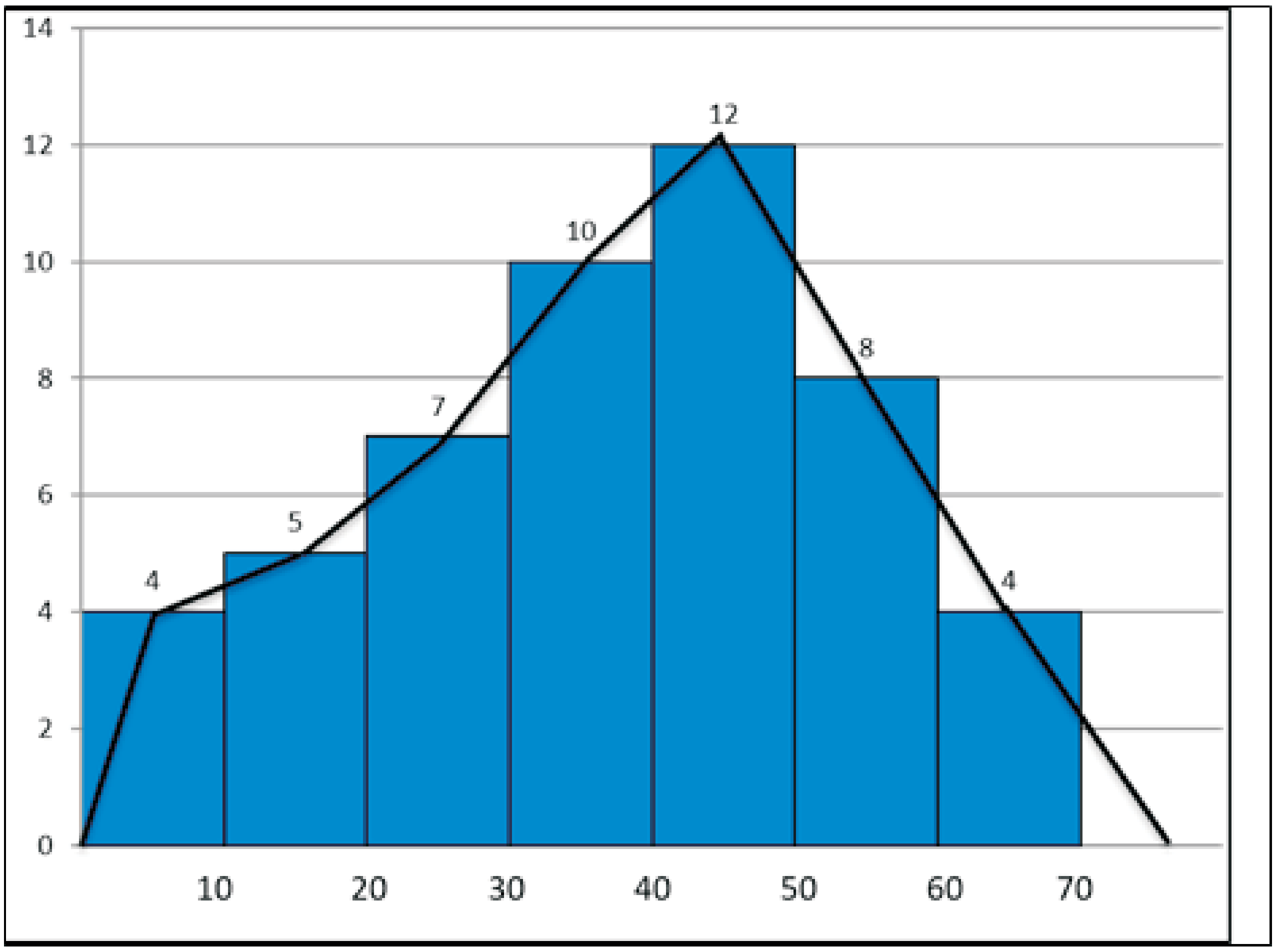
Step 2: Locate the class marks on top horizontal side of each rectangle in histogram

Step 3: Locate two imaginary interval of common size before and after the last class Interval

Step 4: Join all the midpoints, making it a polygon

Example 17: Construct frequency polygon for the following set of data

Class Interval	0-10	10-20	20-30	30-40	40-50	50-60	60-70
frequency	4	5	7	10	12	8	4



How to make choice out of various graphs

- A bar graph is used to indicate the comparison among different categories where independent variable is non-numerical.
- A pie graph is used for comparing parts of a whole. They do not show any changes over time.
- A line graph is used to display those data (independent variable) which continuously changes over the period of time, through unbroken lines.
- A histogram is used for representing the data intervals.
- Frequency Polygons are used to understand the shapes of distribution.
- The histogram and frequency polygon are two different ways to represent the same data set.

Question 2:

Collect multiple set of data and discuss which type of graph representation is most suitable for a given set of data. Is it possible to represent the data graphically in two or more ways?

Question 3:

India's union budget 2020-21 purpose to change following amendment in tax rate. Represent the information using a suitable graph. Justify your choice.

Taxable Income Slab (Rs.)	Current Tax Rates	New Tax Rates
0-2.5 Lakh	Exempt	Exempt
2.5-5 Lakh	5%	Exempt
5-7.5 Lakh	20%	10%
7.5-10 Lakh	20%	15%
10-12.5 Lakh	30%	20%
12.5-15 Lakh	30%	25%
Above 15 Lakh	30%	30%

Source: PRS India

https://www.prsindia.org/parliamenttrack/budgets/union-budget-2020-21-analysis

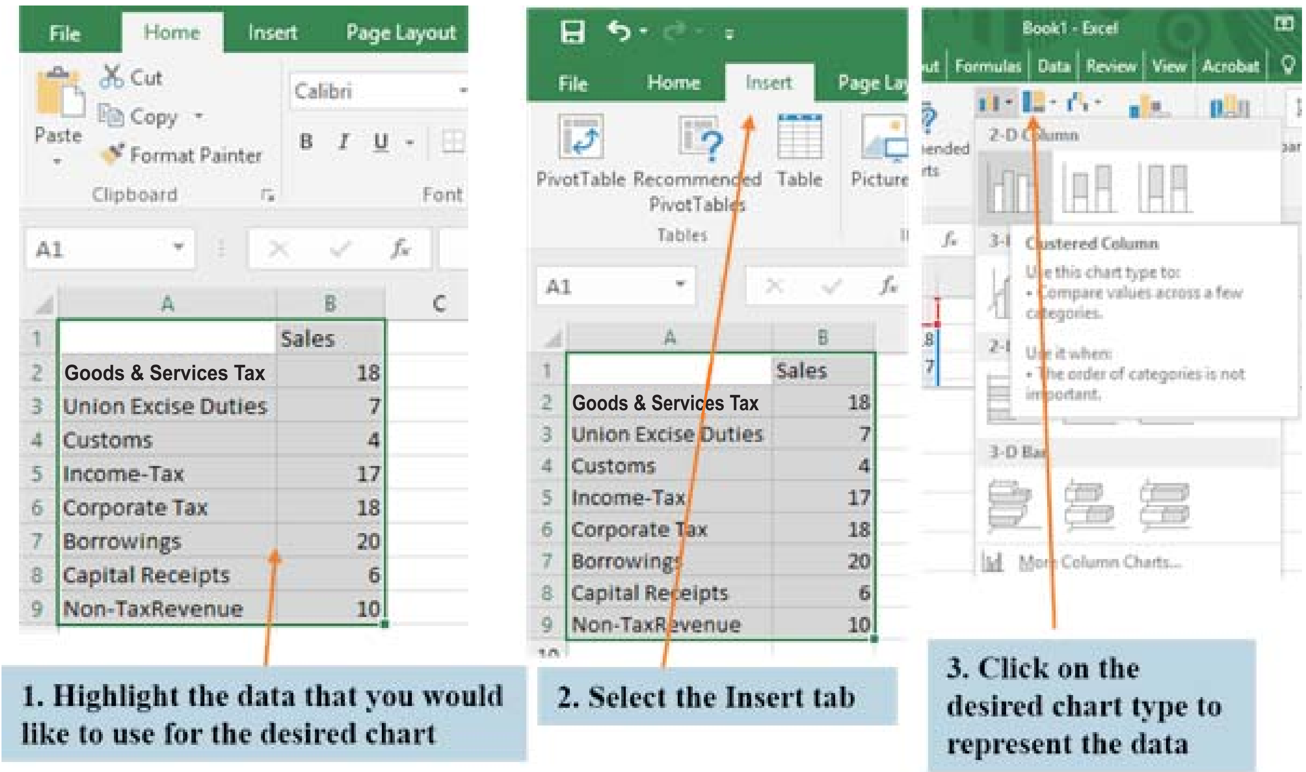
Question 4:

India's union budget 2020-21 fixes Fiscal Responsibility and Budget Management targets FRBM targets for deficits (as % of GDP)

	Actuals 2018-19	Revised 2019-20	Budgeted 2020-21	Target 2021-22	Target 2022-23
Fiscal Deficit	3.4%	3.8%	3.5%	3.3%	3.1%
Revenue Deficit	2.4%	2.4%	2.7%	2.3%	1.9%

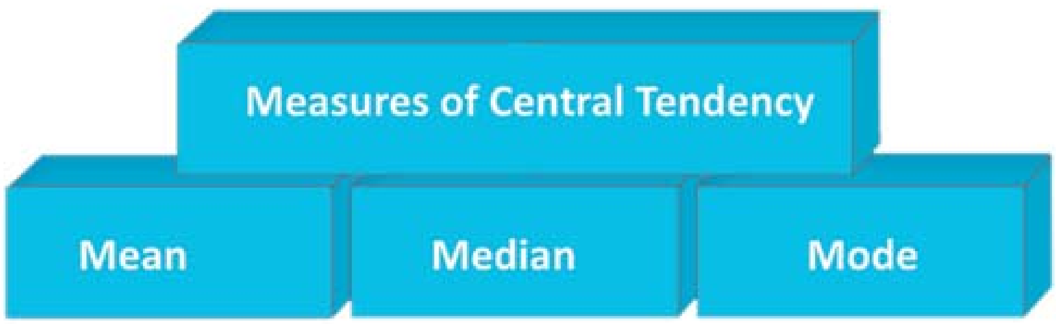
Sources: Medium Term Fiscal Policy Statement, Union Budget 2020-21; PRS.
Represent the information using a suitable graph.

How to create chart in Excel



10.5 Central Tendency

A measure of central tendency is a value that gives the central position of the given data. The most common measures of central tendency are Mean, Median and Mode. It depends on the data and intended purpose for choosing the method of calculating center point.



10.5.1. Mean

The arithmetic mean is usually called the mean or average and is the most common measure of central tendency. It includes every value of data, so it is influenced by outliers which are extreme values. It is denoted by $(\bar{x})$?.

Arithmetic mean of discrete data

$$\text{Arithmetic Mean } (\bar{x}) = \frac{\text{(Sum of all observations)}}{\text{(Number of observations)}}$$

Excel formula for Mean
= AVERAGE(data range)

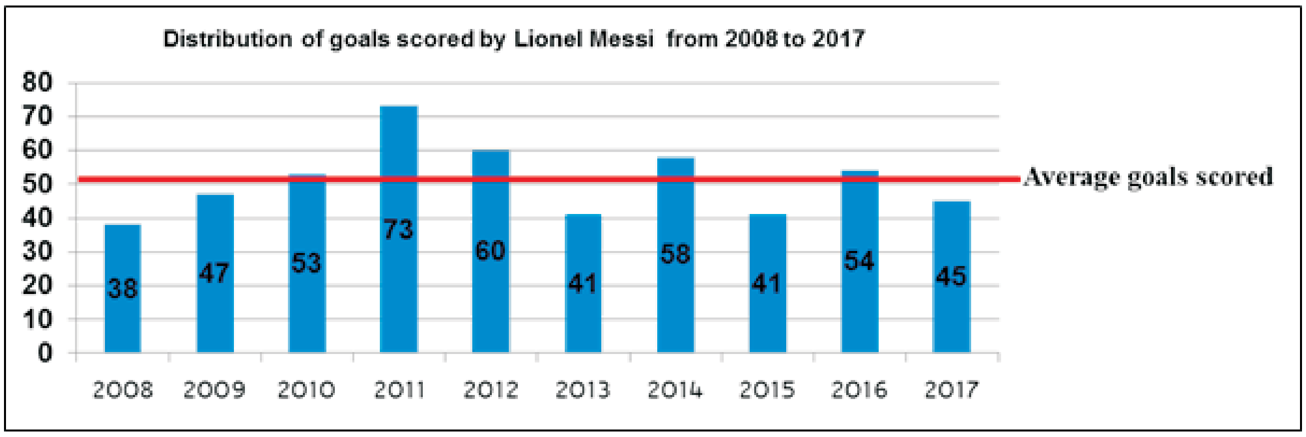
Example 18: Find out the arithmetic mean of goals scored by footballer Lionel Messi from 2008 to 2017 in all competitions;

Year	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017
Goals scored	38	47	53	73	60	41	58	41	54	45

Source: https://messivsronaldo.net/

$$\text{Arithmetic Mean } (\bar{x}) = \frac{38+47+53+73+60+41+58+41+54+45}{10} = \frac{510}{10} = 51 \text{ Goals}$$

This means on an average Lionel Messi scored 51 goals in a year.



Mean of group data

Suppose $x_1 + x_2 + \ldots + x_n$ are the observations and $f_1 + f_2 + \ldots + f_n$ are their respective frequencies then sum of the values of all the observations = $f_{1x1} + f_{2x2} + \ldots + f_{nxn}$, and the number of observations = $f_1 + f_2 + \ldots + f_n$.

The mean $\bar{x}$ of the grouped data is calculated by

i) Direct method → (ii) Assumed mean method → (iii) Step deviation method

The mean obtained by all the three methods is same where assumed mean method and step-deviation method are just simplified forms of the calculating mean by direct method.

(i) **Direct method:**

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} \text{ or } \frac{\sum f_i x_i}{N}$$

Where $\sum f_i x_i$ gives the sum of all the observations and $\sum f_i$ gives the number of observations.

Recall the concept of frequency polygons where we talked about calculating class marks (x_i) by $\frac{\text{Upper Limit + Lower Limt}}{2}$

Example 19: Find the mean marks of the students using direct method.

Class Interval	0-10	10-20	20-30	30-40	40-50
Frequency	7	4	6	3	5

Solution:

Class Interval	Frequency (f_i)	Class mark (x_i)	$f_i x_i$
0 - 10	7	5	35
10 -20	4	15	60
20 -30	6	25	150
30 -40	3	35	105
40 -50	5	45	225
	$\sum f_i = 25$		$\sum f_i x_i = 575$

$(\bar{x})$ Mean = $\frac{\sum f_i x_i}{\sum f_i} = \frac{575}{25} = 23$

(i) **Assumed mean method :**

This method is employed when given data (f_i and x_i) are large. We assume a mean (a) and calculate the deviations (d) from this assumed mean for every observation of data.

$$\bar{x} = a + \frac{\sum f_i d_i}{\sum f_i}$$

where a = assumed mean

$d_i = x_i - a$ deviations from assumed mean

Example 20: Find the mean marks of the students using assumed mean method.

Class Interval	0-20	20-40	40-60	60-80	80-100
Frequency	3	2	5	6	4

Solution:

Class Interval	f_i	Class Mark (x_i)	$d_i = x_i - a$	$f_i d_i$
0-20	3	10	-40	-120
20-40	2	30	-20	-40
40-60	5	50	0	0
60-80	6	70	20	120
80-100	4	90	40	160
	$\sum f_i = 20$			$\sum f_i d_i = 120$

Let assumed mean (a) be 50, then

Mean ( ) = $\bar{x}$ $a+\frac{\sum f_i d_i}{\sum f_i}=50+\frac{120}{20}=56$

(iii) **Step deviation method:** Step deviation method can be used when class-intervals for all the classes in a continuous series are of same magnitude (width) or we can say when deviations ($d_i = x_i - a$) from assumed mean is divisible by a common factor.

$$\bar{x}=a+\frac{\sum f_i u_i}{\sum f_i}\times h$$

where a = assumed mean

h = class width or common factor $u_i = \frac{x_i - a}{h}$

Example 21: Find the mean marks of the students using step deviation method.

Class interval	f_i	Class Mark (x_i)	$d_i = x_i - a$	$u_i = \frac{x_i - a}{h}$	$f_i u_i$
0-10	5	5	-20	-2	-10
10-20	8	15	-10	-1	-8
20-30	15	25	0	0	0
30-40	16	35	10	1	16
40-50	6	45	20	2	12
	$\sum f_i$ =50				$\sum f_i u_i$ =10

Let assumed mean (a) be 25.

Here class width (h) = 10

Therefore, Mean $\bar{x} = a+\frac{\sum f_i u_i}{\sum f_i}\times h = 25+\frac{10}{50}\times 10 = 27$

10.5.2 Median

The median is the number located exactly at the center of data when arranged in an order, either ascending descending. It includes only middle most value of given data, so it is not influenced by the outliers.

Median is a positional average because its value depends upon the position of an item and not on its magnitude and changing the order of data does not changes the median.

Excel formula for Median
= MEDIAN(data range)

Calculating median of ungrouped data:

Step 1: Arrange the data in ascending/descending order

Step 2: Determine the number of observation

Step 3: Find out the middle most observation as follows

- If number of observation is odd, then Median = Middle most in the observation distribution.

  i.e $\left(\frac{n+1}{2}\right)^{th}$ observation.

  For example if n=3, then value of $\left(\frac{3+1}{2}\right)^{th}$ i.e 2^{nd} observation will be the median

- If number of observation is even, then Median = Average of middle observation i.e median is average of $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ observations

- For example if n=4, then mean of $\left(\frac{4}{2}\right)^{th}$ and $\left(\frac{4}{2}+1\right)^{th}$ observations i.e mean of 2^{nd} and 3^{rd} observation will be median.

Example 22: Find median of 14, 3, 1, 7, 10.

Solution:

1	**3**	**7**	**10**	**14**

Median

Step 1: Arrange them in ascending order: 1, 3, 7, 10, 14

Step 2: It has 5 (odd) observations

Step 3: So $\left(\frac{5+1}{2}\right)^{th}$ observation is the median of this data.

i.e 7 is the median of this data.

Median of grouped data:

The median for grouped data is given by the formula:

Median = $l+\left(\dfrac{\frac{n}{2}-cf}{f}\right)\times h$

l = lower limit of median class,

n = number of observations,

cf = cumulative frequency of class preceding the median class,

f = frequency of median class,

h = class size (assuming class size to be equal)

To find the median class, we find the cumulative frequencies of all the classes and locate the class whose cumulative frequency is greater than or closest to. This is called the median class.

Example 23: Following set of data relates to daily wages of persons working in a factory. Compute the median daily wage.

Daily wages (Rs)	200-250	250-300	300-350	350-400	400-450	450-500
No of workers	5	6	8	9	10	12

Solution :

Daily wages (Rs) (Class Interval)	No of workers (f_i)	Cumulative frequency
200 -250	5	5
250 -300	6	11
300 -350	8	19
350 -400	9	28
400 -450	10	38
450 -500	12	50
	$\sum f_i$ =50	

The given set of data is arranged in ascending order where median class is greatest & nearest to $\frac{n^{th}}{2}$ observation (i.e 25[th] item) which lies in 350-400 class interval. By applying the formula of median we get

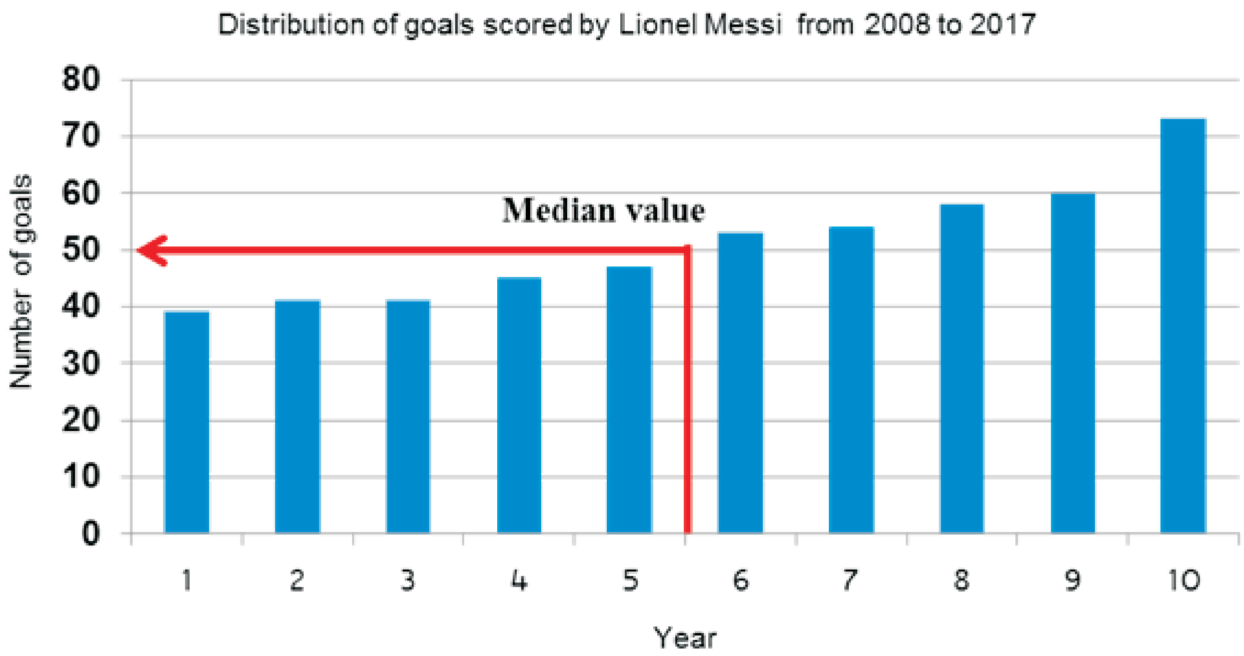
Median = $$l+\left(\frac{\frac{n}{2}-cf}{f}\right)\times h=350+\left(\frac{\frac{50}{2}-19}{50}\right)\times 50=356$$

Therefore, median daily wage is Rs. 356.

This signifies that 50% of the workers receive less than or equal to Rs. 356 and 50% of the workers receive more than or equal to this value.

Note: Geometrically the median (value on x-axis) divides a histogram of data into two parts having equal areas.

Example 24: Observe the graph of data given in earlier example of mean where goals scored by footballer Lionel Messi from 2008 to 2017 in all competitions were given;



Here total number of year are even number, so positional average (median) exactly divides the graph into two parts having equal areas.

10.5.3 Mode

The mode is the value that occurs most frequently in a set of data.There can be more than one mode depending upon the given set of data. For data having one mode the distribution is said to be uni-model, and for two modes the distribution is said to be bi-model.

No mode data: When each value occurs, the same number of times in the data then there is no mode.

Multi-mode data: If two or more values occur the same number of times, then there are two or more modes and the data set is said to be multi-mode.

Excel formula for Mode
= MODE(data range)

Mode of ungrouped data:

When the data has discrete set of values, then mode may be found by simply inspection method and finding the observation having maximum frequency.

Example 25:

- The set of data 1,2,3,3,7,9,12,11,13,13,13, 15 has mode 13 (Uni-model).
- The set of data 4, 6, 8, 12, 15, 16 has no mode.
- The set if data 2, 3, 4, 5, 5, 5, 6, 6, 6, 7, 7, 8 has two modes, 5 and 6 (bimodal).

Mode of grouped data

Mode of continuous series lies in a particular class (modal class). The following method is used in determining mode:

Mode = $l+\left(\frac{f_1-f_0}{2f_1-f_0-f_2}\right)\times h$

A class interval with the maximum frequency is called the modal class.

Where,

l = lower limit of the modal class,

h = class size (assuming class size to be equal)

f1 = frequency of the modal class,

f0 = frequency of the class preceding the modal class,

f2 = frequency of the class succeeding the modal class.

Example 26: Find out mode of the following series:

Class Interval	0-10	10-20	20-30	30-40	40-50
Frequency	8	10	15	7	10

Solution:

Class Interval	Frequency (f_i)
0 - 10	8
10 - 20	10
20 - 30	15
30 - 40	7
40 - 50	10

Here modal class is 20-30, So l=20, f_1 = 15, f_0 = 10, f_2 = 7, h = 10

We know that Mode = $l+\left(\dfrac{f_1-f_0}{2f_1-f_0-f_2}\right)\times h$

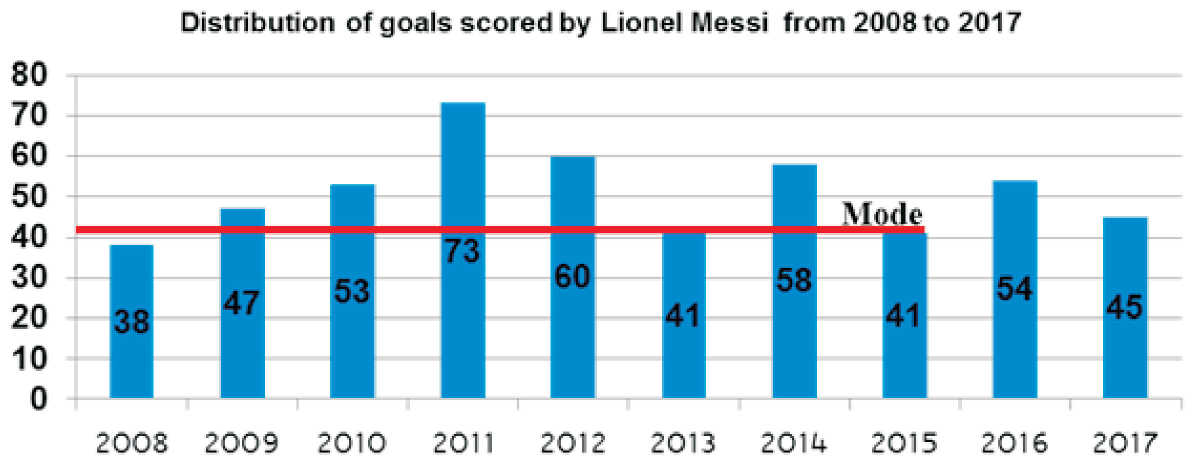
Mode = $20+\left(\dfrac{15-10}{2\times 15-10-7}\right)\times 10=23.48$

Therefore, mode of given series is 23.48

Note: Geometrically In the case of grouped data where a frequency curve has been constructed to fit the data, the mode will be the value (or values) of X corresponding to the maximum point (or points) on the curve.

Example 27: Observe the graph of data given in earlier example of mean where goals scored by footballer Lionel Messi from 2008 to 2017 in all competitions were given;

Year	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017
Goals scored	38	47	53	73	60	41	58	41	54	45



Note:

For uni - model frequency data there exist a relation between mean, median, and mode:

Mean - Mode = 3 (Mean - Median)

10.6 Dispersion

We have already studied that measures of central tendency (mean, median, mode) which explores the central value of a given data set. But in order to make better interpretation from data, we should have an idea about the spread of the data around the central value. So the degree to which data tend to spread around the central value is called dispersion or variation of the data. It measures the variation of the items among themselves and the variation around the central value.

Measures of dispersion / variation in a distribution

The dispersion in a data is measured on the basis of the observations and the types of the measure of central tendency, used there. Several measures of this dispersion are available, the most common are

(i) Range (R)

(ii) Quartile Deviation (QD)

(iii) Mean Deviation (MD)

(iv) Standard Deviation (SD)

10.6.1 Range:

Range is the difference between the highest and lowest value of the given distribution. It provides a quick estimate of the variability of the two distributions. However, the range is calculated only based upon the two extreme values. So at times, it may mislead the variability of data if extremes are rare or unusual.

Range = Maximum value - Minimum value

Coefficient of range = $\dfrac{Max. - Min.}{Max. + Min.}$

Excel formula for range
In Excel the range is calculated using the MIN and MAX functions =MAX(data range) MIN(data range)

Example 28: Find out the range of salaries for staffs of a schools.

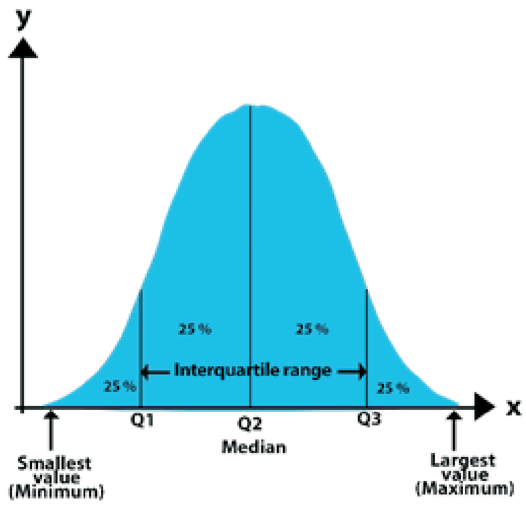
Staff	Salary (Rs.)
Principal:	100000
Senior Teacher	70000
Junior Teacher	45000
Other workers	25000

Range = Maximum value - Minimum value = 100000-25000 = Rs. 75000

Above example make it clear that range is not based on all the values of data, simply calculated by extreme values. Until maximum and minimum values remain intact, any change in other values does not affect range. So in order to have a more meaningful statistic and measure the variability, we use measures called the variance and standard deviation.

10.6.2 Quartile Deviation (QD)

Quartiles separate the original set of data into four equal parts. Each of these parts contains one-quarter (25%) of the data where two halves are less than median and other two greater than median.



- The difference between the third and first quartiles is called the interquartile range (IQR) = Q3 - Q1
- The IQR is sometimes called the middle half and effective in identifying the outliers in data.
- An outlier is any value at least 1.5 IQR above Q3 or below Q1

Quartile deviation (Q.D.) = $\frac{Q_3 - Q_1}{2}$

Quartile deviation of ungrouped data

Step 1: Arrange the values in ascending order and assign serial number to each.

Step2: Determine first quartile (Q1), third quartile (Q3) by following formula

$$Q1 = \frac{1}{4}(n+1)^{th}\ value$$

$$Q3 = \frac{3}{4}(n+1)^{th}\ value$$

Step3: Calculate the Quartile deviation using the formula

Quartile deviation (Q.D.) = $\frac{Q_3 - Q_1}{2}$

Excel formula for quartiles	
Q_1	= PERCENTILE(Data range, 0.25)
Q_2	=MEDIAN(Data range)
Q_3	= PERCENTILE(Data range, 0.75)

Example 29: Calculate the Quartile deviation of following observations:

15, 20, 22, 28, 35, 27, 44, 48, 50, 55, and 60

Solution:

Step 1: Given data: 15, 20, 22, 28, 35, 27, 44, 48, 50, 55, 60

Step 2: Q1 $=\frac{1}{4}(n+1)^{th}$ value $=\frac{1}{4}(11+1)^{th}$ = 3rd value = 22

Similarly, Q3 $=\frac{3}{4}(n+1)^{th}$ value $=\frac{3}{4}(11+1)^{th}$ = 9th value = 50

Step 3: Quartile deviation (Q.D.) = $\frac{Q_3-Q_1}{2} = \frac{50-22}{2} = 14$

Therefore, Quartile deviation of given observations is 14.

Quartile deviation of grouped data

Step 1: Calculate cumulative frequencies corresponding to each value.

Step 2: Determine first quartile (Q1), third quartile (Q3) by following formula

$$Q1 = l + \left(\frac{\frac{n}{4} - cf}{f}\right) \times h$$

$$Q3 = l + \left(\frac{\frac{3n}{4} - cf}{f}\right) \times h$$

l = lower limit of class containing the score

n = total number of values

cf = cumulative frequency of preceding class

f = frequency of class interval

h = class size

where Q1 is the size of $\left(\frac{n}{4}\right)^{th}$ value and Q3 is the size of $\left(\frac{3n}{4}\right)^{th}$ value

Step 3: Calculate the Quartile deviation using the formula

Quartile deviation (Q.D.) = $\frac{Q_3-Q_1}{2}$

Example 30: Calculation the Quartile Deviation from following frequency distribution:

Class Interval	30-34	35-39	40-44	45-49	50-54	55-59	60-64	65-69	70-74	75-79	80-84
Frequency	1	2	6	7	9	11	8	7	5	3	1

Solution:

Class Interval	Frequencies (f)	Cumulative Frequencies (cf)
30-34	1	1
35-39	2	3
40-44	6	9
45-49	7	16
50-54	9	25
55-59	11	36
60-64	8	44
65-69	7	51
70-74	5	56
75-79	3	59
80-84	1	60
	N=60	

We know, Q1 is the size of $\left(\frac{n}{4}\right)^{th}$ value and Q3 is the size of $\left(\frac{3n}{4}\right)^{th}$ value

$\longrightarrow$ Q1 is the size of $\left(\frac{60}{4}\right)^{th}$ value = 15th value. The class containing the 15th value is 45-49.

$$Q1 = l + \left(\frac{\frac{n}{4} - cf}{f}\right) \times h \quad = 44.5 + \left(\frac{\frac{60}{4} - 9}{7}\right) \times 5 = 48.79$$

Q3 is the size of $\left(\frac{180}{4}\right)^{th}$ value = 45th value. The class containing the 45th value is 65-69.

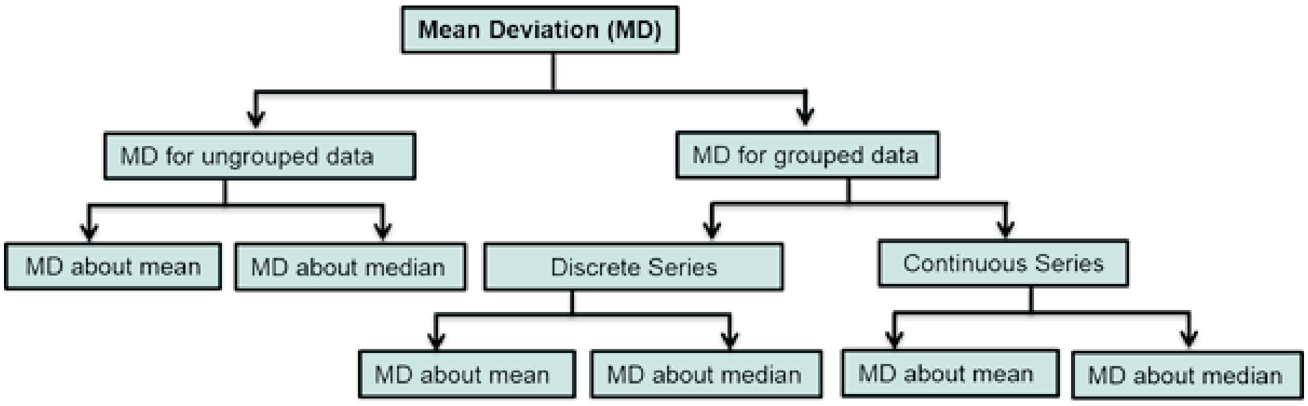
Calculating Q3: = $l + \left(\frac{\frac{3n}{4} - cf}{f}\right) \times h = 64.5 + \left(\frac{45-44}{7}\right) \times 5 = 65.21$

Quartile deviation (Q) = (Q.D.) = $\frac{Q_3 - Q_1}{2} = \frac{65.21 - 48.79}{2} = 8.21$

Therefore quartile deviation of given data is 8.21

In both individual and discrete series, Q_1 is the size of $\frac{1}{4}(n+1)^{th}$ value, but in a continuous distribution, it is the size of $\left(\frac{n}{4}\right)^{th}$ value. Similarly, for Q_3 and median (Q_2) also, n is used in place of n+1.

10.6.3 Mean Deviation (MD)



A. Mean Deviation for ungrouped data:

Let n observations be $X_1, X_2, X_3, X_4, X_5 \ldots\ldots\ldots\ldots\ldots\ldots X_n$ then

(I) Mean deviation about mean

M.D. $(\bar{x}) = \frac{1}{N}\sum_{i=1}^{n}|x_i - \bar{x}|$ where $\bar{x}$ = mean

(II) Mean deviation about median

M.D. $(median) = \frac{1}{N}\sum_{i=1}^{n}|x_i - M|$ where M = median

Example 31: Find the mean deviation about the mean and median for the data: 4,3,2,5,7,6,8.

Solution: The mean of data, $(\bar{x}) = \frac{4+3+2+5+7+6+8}{7} = 5$

The deviation of the respective observation from the mean $(\bar{x})$ ie $x_i - \bar{x}$ are

4-5, 3-5, 2-5, 5-5, 7-5, 6-5, 8-5 or -1, -2, -3, 0, 2, 1, 3

The absolute values of deviations i.e $|x_i - \bar{x}|$ are 1, 2, 3, 0, 2, 1, 3

So, mean deviation mean, M.D. $(\bar{x}) = \frac{1}{N}\sum_{i=1}^{n}|x_i - \bar{x}|, = \frac{1+2+3+0+2+1+3}{7} = 1.71$

Now, Median $= (\frac{7+1}{2})^{th} =$ 4^{th} observation = 5

The deviation of the respective observation from the median, i.e. x_1-median are

4-5, 3-5, 2-5, 5-5, 7-5, 6-5, 8-5 or -1, -2, -3, 0, 2, 1, 3

The absolute values of deviations i.e $|x_i - M|$ are 1, 2, 3, 0, 2, 1, 3

So mean deviation about median,

M.D. $(median) = \frac{1}{N}\sum_{i=1}^{n}|x_i - M| = \frac{1+2+3+0+2+1+3}{7} = 1.71$

B. Mean deviation for grouped data

Let the given data consist of n distinct values $x_1, x_2, x_3, x_4, x_5 ....x_n$

having frequencies $f_1, f_2, f_3, f_4, f_5 ....f_n$ respectively.

We know that data can be grouped into two ways:

(a) Discrete frequency distribution

(b) Continuous frequency distribution

(a) Discrete frequency distribution

(i) Mean deviation about mean

M.D. $(\bar{x}) = \frac{1}{N}\sum_{i=1}^{n} f_i |x_i - \bar{x}|$ where $\bar{x} = $ mean

Example 32: Suppose a class of 25 students conducted a quiz and grades obtained are given in following table. Find the mean deviation about mean of data.

Grade	5	10	15	20	25
Number of students	7	4	6	3	5

Solution:

x_i	Frequency (f_i)	$f_i x_i$	$\lvert x_i - \bar{x}\rvert$	$f_i \lvert x_i - \bar{x}\rvert$
5	7	35	9	63
10	4	40	4	16
15	6	90	1	6
20	3	60	6	18
25	5	125	11	55
	$\sum_{i=1}^{n} f_i = 25$	$\sum_{i=1}^{n} f_i x_i = 350$		$\sum_{i=1}^{n} f_i \lvert x_i - \bar{x}\rvert = 158$

$$Mean = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{350}{25} = 14$$

$$\text{M.D.}(\bar{x}) = \frac{1}{N}\sum_{i=1}^{n} f_i \left|x_i - \bar{x}\right| = \frac{158}{25} = 6.32$$

(ii) Mean deviation about median

M.D. (median) $= \frac{1}{N}\sum_{i=1}^{n} f_i \left|x_i - M\right|$, where M = median

Example 33: Find out the mean deviation about the median for following data.

Height (cm)	147	148	150	152	155
Number of students	8	12	15	10	5

Solution:

Height x_i	Frequency (f_i)	Cumulative frequency (cf)	$\left|x_i - M\right|$	$f_i\left|x_i - M\right|$
147	8	8	3	24
148	12	20	2	24
150	15	35	0	0
152	10	45	2	20
155	5	50	5	25
	50			$\sum_{i=1}^{n} f_i\left|x_i - M\right| = 93$

$$Median(M) = \frac{25^{th} + 26^{th}\ observations}{2} = \frac{150+150}{2} = 150$$

$$M.D.(median) = \frac{1}{N}\sum_{i=1}^{n} f_i\left|x_i - M\right| = \frac{93}{50} = 1.86\ cm$$

(b) Continuous frequency distribution

A continuous frequency distribution is a series in which the data are classified into different class-intervals without gaps alongwith their respective frequencies. Recall the process of finding the mean and median for continuous frequency distribution and apply the same for putting the values of mean and median in calculating the mean deviation.

(i) **Mean deviation about mean**

$\text{M.D.}(\bar{x}) = \frac{1}{N}\sum_{i=1}^{n} f_i \left| x_i - \bar{x} \right|$,where you find out the mean ($\bar{x}$) by methods like short cut, assumed mean or step deviation method as discussed earlier.

(ii) **Mean deviation about median**

$$\text{M.D.}\,(median) = \frac{1}{N}\sum_{i=1}^{n} f_i \left| x_i - M \right|$$

where M (Median)= $l + \left(\frac{\frac{n}{2} - cf}{f} \right) \times h$

Example 34: Find the mean deviation about mean of following data.

Grade	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Number of students	2	3	8	14	8	3	2

Solution:

Marks obtained	Frequency (f_i)	Mid Points (x_i)	$f_i x_i$	$\left\| x_i - \bar{x} \right\|$	$f_i \left\| x_i - \bar{x} \right\|$
10-20	2	15	30	30	60
20-30	3	25	75	20	60
30-40	8	35	280	10	80
40-50	14	45	630	0	0
50-60	8	55	440	10	80
60-70	3	65	195	20	60
70-80	2	75	150	30	60
	$\sum_{i=1}^{7} f_i = 40$		1800		400

$$\sum_{i=1}^{7} f_i x_i = 1800, \text{ and } \sum_{i=1}^{7} f_i \left| x_i - \bar{x} \right| = 400$$

Therefore, mean deviation about mean, M.D. $(\bar{x}) = \frac{1}{N}\sum_{i=1}^{n} f_i \left| x_i - \bar{x} \right| = \frac{1}{40} \times 400 = 10$

The procedure to calculate Mean Deviation about median is the exactly same as M.D. about Mean, except that deviations are to be taken from the median as shown in next example.

Example 35: Find the mean deviation about median of following data.

Class interval	20-30	30-40	40-60	60-80	80-90
Frequency	5	10	20	9	6

Solution :

Class interval (CI)	Frequency (f_i)	Mid Points (x_i)	Cf	$\lvert x_i - M\rvert$	$f_i\lvert x_i - M\rvert$
20-30	5	25	5	25	125
30-40	10	35	15	15	150
40-60	20	50	35	0	0
60-80	9	70	44	20	180
80-90	6	85	50	35	210
	$\sum_{i=1}^{7} f_i = 50$				665

$$Median = l + \left(\frac{\frac{n}{2} - cf}{f}\right) \times h = 40 + \frac{(25-15)}{\cancel{20}} \times \cancel{20} = 50$$

$$\text{M.D.}(Median) = \frac{1}{N}\sum_{i=1}^{n} f_i \lvert x_i - M\rvert = \frac{665}{50} = 13.3$$

Practical example of mean deviation: Consider that Govt. is planning to build a school in a city having population N. The suitable location of school is to be decided such that average distance covered by students is minimized.

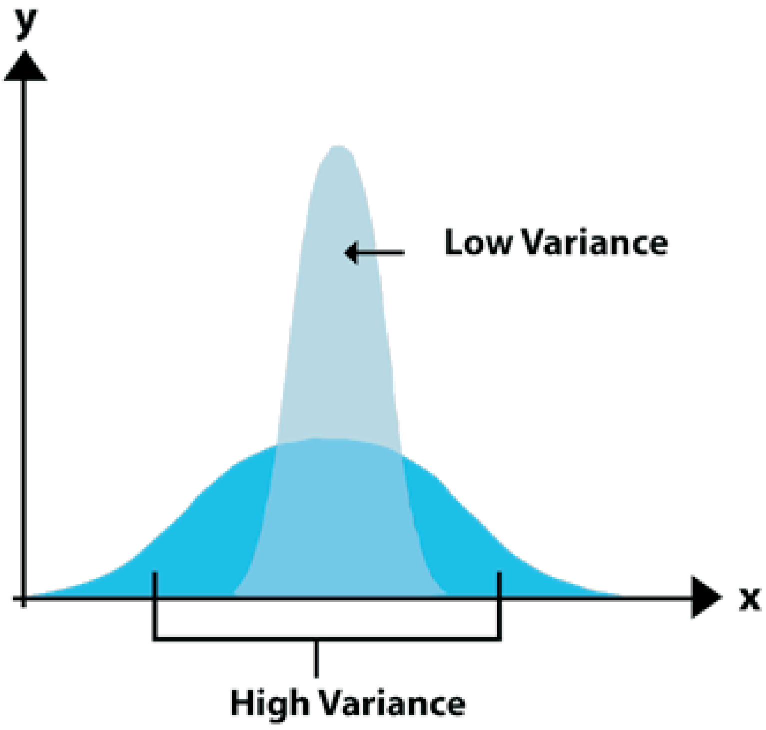
Let X_i be the distance of each student from any specific location point and 'a' be the distance of proposed site of school from same specific location. The distance covered by each student is $\lvert X_i - a\rvert$ and average distance is going to be MD. Now locate a point such that sum of all the $\lvert X - a\rvert$ is minimized. Basically we are minimizing the value of MD.

10.7 Variance

The variance is the average of the squares of the deviation of each value from mean and denoted by'σ'(read as sigma square). Therefore, the variance of n observations $x_1, x_2, x_3 ....x_n$ is given by;

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$$

So far discussed measures of dispersion (range, mean deviation, quartile deviation) describes only the measure of the theoretical averages but does not tell anything about the spread of the distribution. The variance combines all the values in a data set to produce a measure of the spread. Basically it is the arithmetic mean of the squared differences between each value and the mean value.



Example 36: Find the variance for the data set 10, 60, 50, 30, 40, 20.

Solution: The mean of data $(\bar{x}) = \frac{10+60+50+30+40+20}{6} = \frac{210}{6} = 35$

Variance $(\sigma^2) = \frac{1}{N}\sum_{i=1}^{n}(x_i - \bar{x})^2$

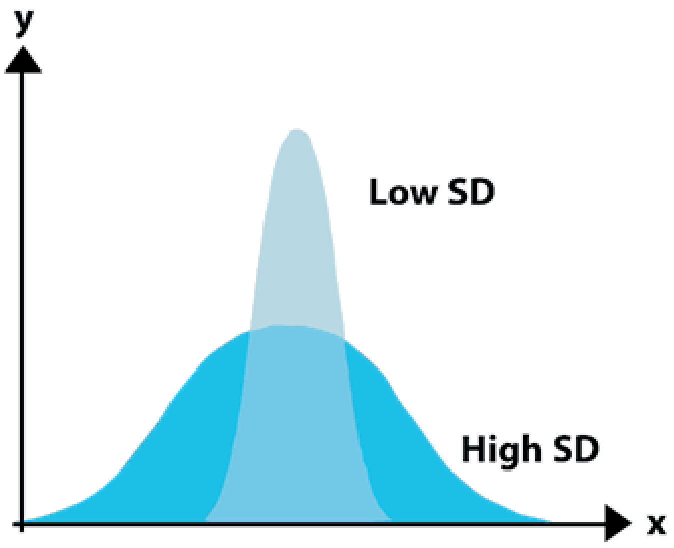
$$= \frac{(10-35)^2 + (60-35)^2 + (50-35)^2 + (30-35)^2 + (40-35)^2 + (20-35)^2}{6}$$

$$= \frac{1750}{6} = 291.7$$

10.7.1 Standard Deviation (SD)

Standard deviation (SD) measures variation (spread) in the data by finding the distances (deviations) between each data value and the mean (average).

The standard deviation (SD) gives an idea of the shape of the distribution and indicates how much variation there is from the mean and considered preferred measure of variability.



- Low SD indicates that the data points tend to be very close to the mean., thus very reliable
- High SD indicates that the data is spread out over a large range of values (large variance between data and average), thus not reliable.

The standard deviation is the square root of the variance and given by

Standard Deviation $(\sigma) = \sqrt{\frac{1}{N}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}$

Excel formula for S.D.
= STDEV(data range)

Example 37: Find the standard deviation of data set 9, 3, 8, 8, 9, 8, 9, 18.

The mean of data $\left(\bar{x}\right) = \frac{9+3+8+8+9+8+9+18}{8} = \frac{72}{8} = 9$

Now, SD $(\sigma) = \sqrt{\frac{1}{N}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}$

$$(\sigma) = \sqrt{\frac{(9-9)^2+(3-9)^2+(8-9)^2+(8-9)^2+(9-9)^2+(8-9)^2+(9-9)^2+(18-9)^2}{8}}$$

$$(\sigma) = \sqrt{15} = 3.87$$

Properties of Standard Deviation:

- Standard deviation is always positive.
- The outliers influence standard deviation. A single outlier can increase the standard deviation.

- For a data set having all the values same, the standard deviation is zero because each value is equal to the mean.

Effect of constant changes to the original data:

- If you add/subtract a constant value k to all the values, the arithmetic mean increases/decreases by k but the standard deviation remains the same.
- If you multiply/divide all the values by a constant value k, both the arithmetic mean and the standard deviation are multiplied/divided by k.

Standard Deviation of discrete frequency distribution

Let the given data consist of n distinct values $x_1, x_2, x_3, x_4, x_5 ....x_n$ having frequencies $f_1, f_2, f_3, f_4, f_5 ....f_n$ respectively. Then

$(\sigma) = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i \left(x_i - \bar{x}\right)^2}$ where $\bar{x}$ = mean and $N = \sum_{i=1}^{n} f_i$

Example 38: Thirty students were asked that how many days they require to revise the chapter of descriptive statistics. Their responses were: 4, 5, 6, 5, 3, 2, 8, 0, 4, 6, 7, 8, 4, 5, 7, 9, 8, 6, 7, 5, 5, 4, 2, 1 , 9, 3, 3, 4, 6, 4. Find out the standard deviation for this data.

No. of Days (x_i)	Frequency (f_i)	$(f_i x_i)$	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
0	1	0	-5	25	25
1	1	1	-4	16	16
2	2	4	-3	9	18
3	3	9	-2	4	12
4	6	24	-1	1	6
5	5	25	0	0	0
6	4	24	1	1	4
7	3	21	2	4	12
8	3	24	3	9	27
9	2	18	4	16	32
	$\sum f_i = 30$	$\sum f_i x_i = 150$			$\sum f_i(x_i - \bar{x})^2 = 152$

means $\bar{x} = \dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{150}{30} = 5$

Standard Deviation $(\sigma) = \sqrt{\dfrac{1}{N}\sum_{i=1}^{n} f_i\left(x_i - \bar{x}\right)^2} = \sqrt{\dfrac{152}{30}} = 2.25$

Standard Deviation of continuous frequency distribution:

The given continuous frequency distribution can be represented as a discrete frequency distribution by replacing each class by its mid-point. Then, the standard deviation is calculated by the technique adopted in the case of a discrete frequency distribution, and obtained by the formula

$(\sigma) = \sqrt{\dfrac{1}{N}\sum_{i=1}^{n} f_i\left(x_i - \bar{x}\right)^2}$ where mean and $N = \sum_{i=1}^{n} f_i$

Example 39: In a test of 50 students, marks secured by students is given in the following table. Calculate the standard deviation of scores.

Height (inch)	10-20	20-30	30-40	30-40	40-50
Number of students	5	8	15	16	6

Solution:

Class	Frequency (f_i)	Mid points (x_i)	$u_i = \dfrac{x_i - a}{h}$	$f_i u_i$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
0-10	5	5	-2	-10	484	2420
10-20	8	15	-1	-8	144	1152
20-30	15	25	0	0	4	60
30-40	16	35	1	16	64	1024
40-50	6	45	2	12	324	1944
	$\sum f_i = 50$			$\sum f_i u_i = 10$		$\sum f_i(x_i - \bar{x})^2 = 6600$

Mean $\bar{x} = a + \dfrac{\sum f_i u_i}{\sum f_i} \times h = 25 + \dfrac{10}{50} \times 10 = 27$

$(\sigma) = \sqrt{\dfrac{1}{N}\sum_{i=1}^{n} f_i\left(x_i - \bar{x}\right)^2} = \sqrt{\dfrac{6600}{50}} = \sqrt{132} = 11.48$

Short cut method of Standard Deviation S.D.:

When values of x_i in a discrete distribution or mid points of x_i in continuous distribution are large then we can apply following step deviation method

$$(\sigma) = \frac{h}{N}\sqrt{N\sum_{i=1}^{n} f_i y_i^2 - \left(\sum_{i=1}^{n} f_i y_i\right)^2}$$

Where , $y_i = \frac{x_i - a}{h}$ a=Assumed mean and h = height of class

Example 40: In a survey, height of 100 students was observed and is given in the following table. Calculate the standard deviation of values.

Height (inch)	60-62	63-65	66-68	69-71	72-74
Number of students	5	18	42	27	8

Solution:

Class	Mid-point (x_i)	Frequency (f_i)	$y_i = \frac{x_i - a}{h}$	y_i^2	$(f_i y_i)$	$(f_i y_i^2)$
60-62	61	5	-2	4	-10	20
63-65	64	18	-1	1	-18	18
66-68	67	42	0	0	0	0
69-71	70	27	1	1	27	27
72-74	73	8	2	4	16	32
		100			$\sum f_i y_i = 15$	$\sum f_i y_i^2 = 97$

Let assumed mean (a) = 67

We know $(\sigma) = \frac{h}{N}\sqrt{N\sum_{i=1}^{n} f_i y_i^2 - \left(\sum_{i=1}^{n} f_i y_i\right)^2}$

Where $y_i = \frac{x_i - a}{h}$ A=Assumed mean and h = height of class

So, $(\sigma) = \frac{3}{100}\sqrt{100(97) - (15)^2} = 2.92$ inch

Practical example of mean deviation: Suppose a weather forecasting agency is analyzing the high temperatures predicted for a series of days versus actual high temperature recorded on each of the day. A low value of SD would show reliable weather forecasting.

Check your Progress:

Questions 5: Find the standard deviation of the data 3, 6, 2, 1, 7, 5.

Question 6: Calculate standard deviation for the following set of scores: 40, 38, 42, 60, 72, 54.

Question 7: By multiplying each of the numbers 3, 6, 2, 1, 7, and 5 by 2 and then adding 5, we obtain the set 11, 17, 9, 7, 19, 15. What is the relationship between the standard deviations and the means for the two sets?

Use of measures of deviation

- Range is used when scores are spread broadly and need rapid, crude and at a glance measure.
- Quartile deviation is used when median has been used as a measure of central tendency.
- Standard Deviation (SD) is the most consistent and stable index of variability and used when mean has been used as measure of central tendency.

10.8 Skewness

Skewness denotes the degree of asymmetry in the data i.e tendency of a distribution to depart from symmetry. Skewness can be positive, negative or zero. The aim of measuring skewness is to know the direction of variation from an average and to compare the frequency distribution and shape of their curve.

Excel formula for Skewness
= SKEW(data range)

(i) Positive Skewness

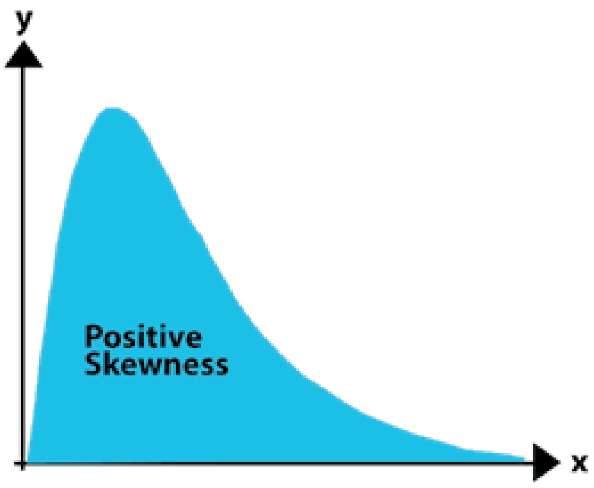
When skewness is positive, distribution is called "positively skew"

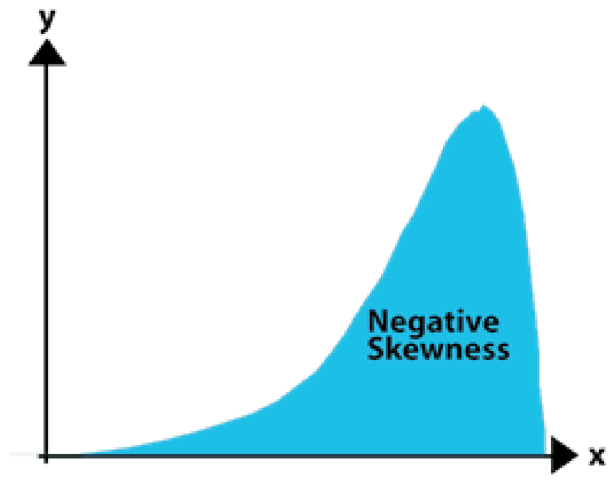
In this case, distribution has long tail on right and Mean > Median > Mode

(ii) Negative Skewness

When skewness is negative, distribution is called "negatively skew"

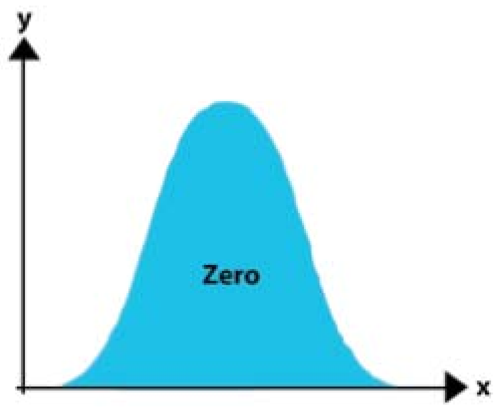
In this case, distribution has long tail on left and Mean < Median < Mode





(iii) **Zero Skewness**

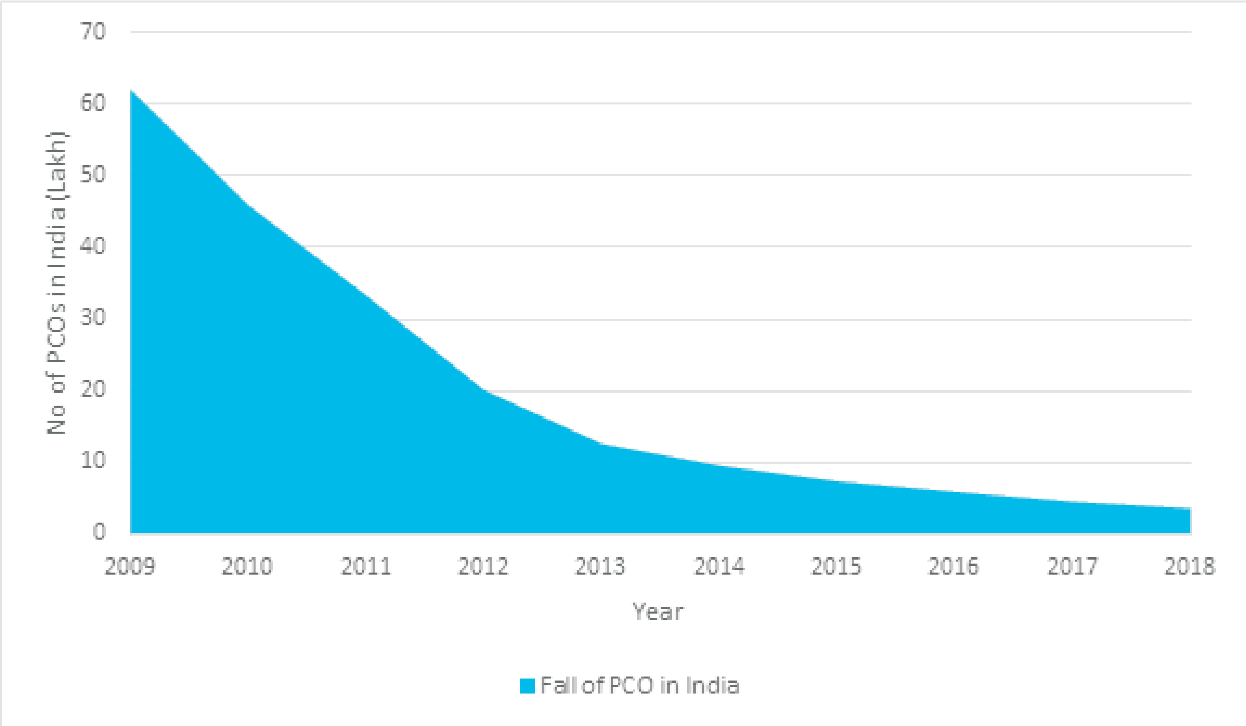
When skewness is zero, distribution is called "symmetrical"



Example 41: In the early years of the telecom sector, PCOs (Public Call Offices) became ubiquitous in India. However, spread of mobile services declined its number in subsequent years. Telecom Statistics India report (2018) gives following figures of PCOs in recent years

Year	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
No of PCOs (in Lakh)	62	45.90	33.30	20.10	12.60	9.57	7.37	5.89	4.52	3.60

Source: Telecom Statistics India report (2018) (TRAI records)
https://dot.gov.in/sites/default/files/statistical%20Bulletin-2018.pdf



By plotting the given data, we get to know that distribution is positively skewed as it has long tail on right.

Measure of skewness: Coefficient of skewness

(i) Pearson's 1st coefficient of skewness $= \dfrac{Mean - Mode}{S.D.}$

(ii) Pearson's 2nd coefficient of skewness $= \dfrac{3(Mean - Median)}{S.D.}$

(iii) Quartile coefficient of skewness $= \dfrac{Q_3 + Q_1 - 2Q_2}{S.D.}$

(iv) Percentile coefficient of skewness $= \dfrac{P_{90} + P_{10} - 2Median}{P_{90} - P_{10}}$

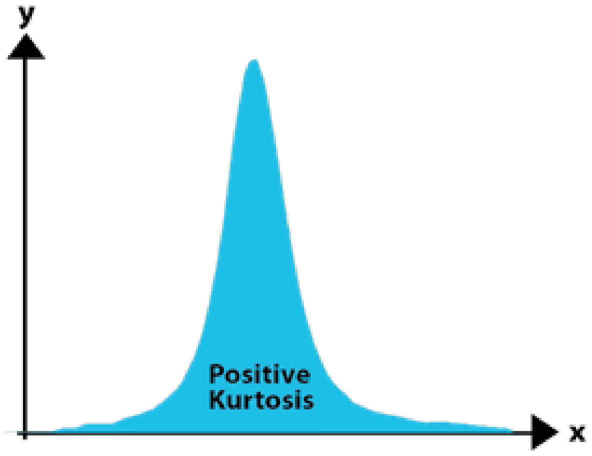
(v) Decile coefficient of skewness $= \dfrac{D_9 + D_1 - 2Median}{D_9 - D_1}$

10.9 Kurtosis

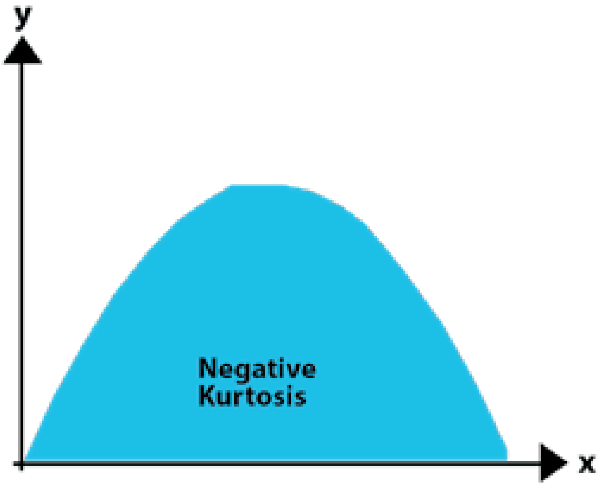
Kurtosis denotes the degree of 'peakedness' of frequency curve. It is used to specify the frequency curve as regards the sharpness of its peak. Kurtosis can be positive, negative or zero.

Excel formula for Kurtosis
= KURT(data range)

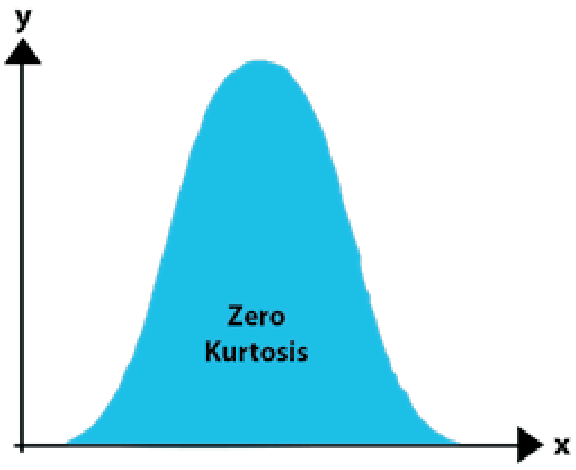
(i) **Positive Kurtosis (or leptokurtic):** A frequency distribution of data set having a **sharp peak** (heavy tailed) / outliers.



(a) **Negative Kurtosis (or platykurtic) :** A frequency distribution of data set having a blunt peak (light tailed)



(ii) **Zero Kurtosis (or mesokurtic):** A frequency distribution of data set having a moderate peak. This means the kurtosis is same as the normal distribution.



10.10 Measures of Position

Measures of position identifies the position of a value, relative to other values in a set of data. The most common measures are of position are percentiles, quartiles and deciles. Here we shall study only percentiles and quartiles.

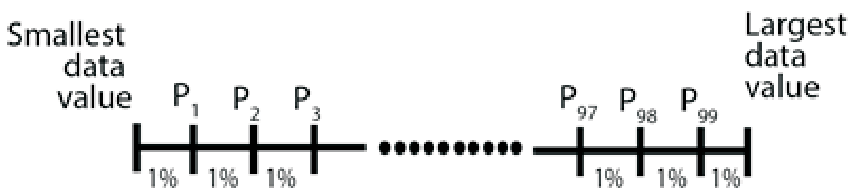
10.10.1 Percentile Rank

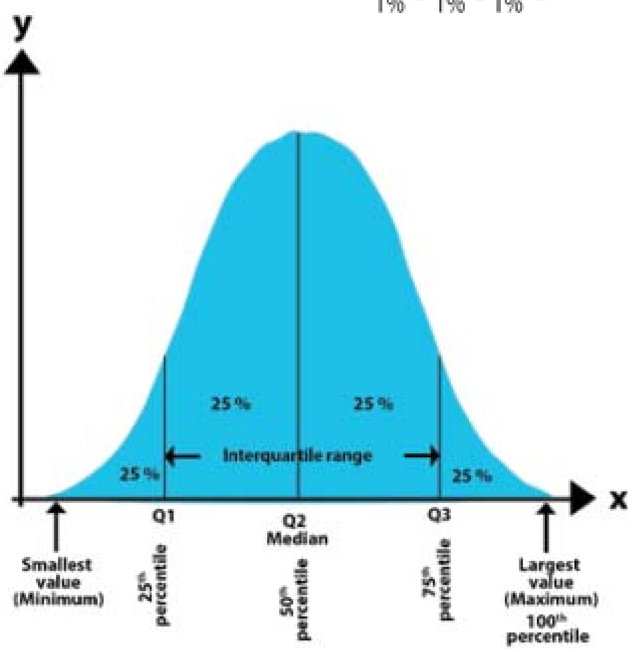
Percentile: Percentile indicate the position of an individual in a group by a number wherea certain percentage of scores fall below that particular number. The number below which P% of the values fall is called the Pth percentile

Suppose you scored 70 marks out of 80 in your Mathematics paper. This figure becomes more meaningful if you get to know that your score is 80th percentile, then it means you scored better than the 80% of students who took the same paper.

Now a days, percentile is very commonly used for declaring the results where number of candidates appeared is very large. It reflect your individual scoring in comparison to all.

Percentiles are denoted by P_1, P_2, P_3...........P_{99} and divide the distribution into 100 groups.





NOTE

- 25th percentile = First quartile (Q1)
- 50th percentile = Median = Second quartile (Q2)
- 75th percentile = Third quartile (Q3)
- Percentiles are not the same as percentages.

For a value X in a data set, X is the Pth percentile of the data if the percentage of the data that are less than or equal to X is P. The number P is the percentile rank of X.

Percentile rank: The percentile rank of a value is the percentage of entries smaller than that value. It is a measure of relative performance.

- 1st percentile = number such that 1% of the entries are smaller than the number, and 99% are larger
- 25th percentile = number such that 25% of the entries are smaller than the number, and 75% are larger
- The percentile rank of the highest secured marks is 100%
- The percentile rank of the median secured marks is 50%

NOTE: A percentile is a number.
A percentile rank is a percentage.

Excel formula for percentile (P_k)
= PERCENTILE(Data range, k in decimal form)

Percentile rank (PR) of ungrouped data (individual series):

Step 1: Put data in ascending order

Step 2: Find rank position (serial number) of individual score

Step 3: Percentile rank (PR) for a particular value Xi of data set is given by

PR = $100-\left(\frac{100R-50}{N}\right)$

where R = Rank position of item within the distribution and N = Total number of items

Example 42: Suppose your class teacher conducted a 20 marks unit test. You scored 12 marks and others obtained 20, 10, 18, 15, 6, 8, 2, 3, 5, 17, 19, 20. Find out your percentile rank of your score.

Solution: Given data = 20, 10 18, 15, 6, 8, 2, 3, 5, 17, 19, 20, 12

Step 1: Put the given data in ascending order

2,3,5,6,8,10,12,15,17,17,18,19,20

We know that Percentile rank (PR) = $100 - \left(\frac{100R - 50}{N}\right)$

Here, score of 12 lies at 7th rank, so R = 7 & N = 13

PR = $100 - \left(\frac{100 \times 7 - 50}{13}\right)$

Hence, your score of 12 did better than 50% of the whole class.

Percentile ranks of other values can be find out by simply changing their respective rank positions.

Percentile rank of grouped data:

Step 1: Find cumulative frequency and cumulative frequency percentage of data

Step 2: Find the class interval, lower limit of class containing the score whose percentile rank is required, cf of preceding class, class size and total frequencies (i.e. N),

Step 3: Calculate x,(number of points required to be added to the exact value of the lower limit of class-interval in order to reach the score for which the PR is to be calculated)

i.e. x = Score whose percentile is required - Lower limit of class interval

Step 4: Percentile rank (PR) for a particular value Xi of data set is given by

PR = $\frac{100}{N}\left(cf + \frac{x}{h} \times f\right)$

Example 43: For the following frequency distribution, compute the percentile rank corresponding to the score of 66.

Class Interval	93-97	88-92	83-87	78-82	73-77	68-72	63-67	58-62	53-57	48-52
Frequency	4	7	5	8	3	6	7	10	5	4

Solution :

Class Interval	f	cf	Percentage of cf
93-97	4	59	100
88-92	7	55	93.22
83-87	5	48	81.36
78-82	8	43	72.88
73-77	3	35	59.32
68-72	6	32	54.24
63-67	7	26	44.07
58-62	10	19	32.30
53-57	5	9	15.25
48-52	4	4	6.78

Step 1: Find cumulative frequency and cumulative frequency percentage of data

Step 2: For score 66, Class interval is 63-67, l = 62.5 (by making it continuous),

Class size = 5 and N=59

Step 3: x = 66-62.5 = 3.5, f = 7, cf = 19

PR (for the score of 66) = $\frac{100}{59}\left(19+\frac{3.5}{5}\times 7\right)=40.50$

Thereby percentile rank of score 66 is 40.50

Check your Progress:

Question 8: Differentiate between percentile and Percentile rank.

Question 9: Calculate reasonable value for PR_{70} and PR_{80} for example no. 43 (given in Percentile rank of grouped data).

10.10.2 Quartile Rank

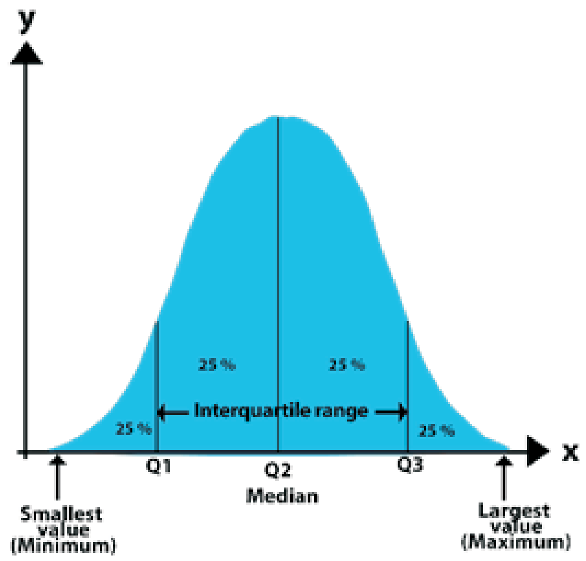
As we have already studied in section 6.6.2 that quartiles breaks the original set of data into four equal parts. Each of these parts contains one-quarter (25%) of the data where two halves are less than median and other two greater than median.

1. First quartile (Q1) contains upto 25% of data

2. Second quartile (Q2) contains 25% - 50% of data (up to the median) of data

3. Third quartile (Q3) contains 50% - 75% (above the median) of data

4. Fourth quartile (Q4) contains 75% - 100% of data



Quartile rank of ungrouped data (individual series):

Step 1: Arrange the values in ascending order and assign serial number to each.

Step2: Determine first quartile (Q1), third quartile (Q3) by following formula

Q1 = $\frac{1}{4}(n+1)^{th}$ Value

Q2 = $\frac{2}{4}(n+1)^{th} = \frac{(n+1)^{th}}{2}$ Value

Q3 = $\frac{3}{4}(n+1)^{th}$ Value

Q4 = $\frac{4}{4}(n+1)^{th}$ Value (This is not dividing the series)

Quartile rank of grouped data:

Step 1: Arrange the values in ascending order.

Step 2: Find out the quartile ranks using the formula

$$Q_1 = l + \left(\frac{\frac{n}{4} - cf}{f}\right) \times h \qquad Q_3 = l + \left(\frac{\frac{3n}{4} - cf}{f}\right) \times h$$

l = lower limit of class containing the score

n = total number of values

cf = cumulative frequency of preceding class

f = frequency of class interval

h = class size

In both individual and discrete series, Q_1 is the size of $\frac{1}{4}(n+1)^{th}$ value, but in a continuous distribution, it is the size of $\left(\frac{n}{4}\right)^{th}$ value. Similarly, for Q_3 and median (Q_2) also, n is used in place of n+1.

- For symmetrical distribution, mean and median are equal and median lies at an equal distance from the two quartiles

  i.e.Q3–Median = Median–Q1

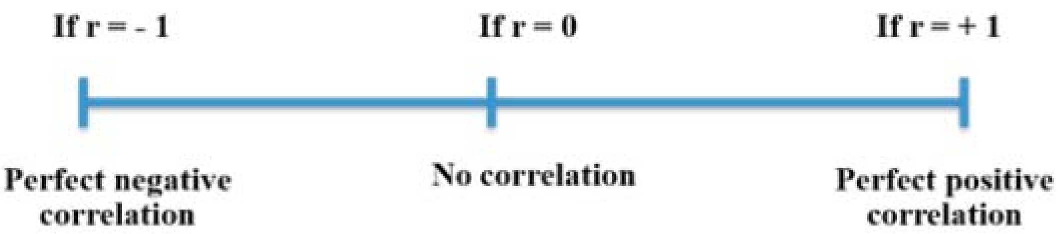
- For non-symmetrical distribution, two possibilities may arise:

  I. Q3–Median > Median–Q1 : (Positive Skewed Curve)

  II. Q3–Median < Median–Q1 : (Negative Skewed Curve)

10.11 Correlation

Correlation refers to a process for establishing relationships between two variables. It gives the extent of relation between two variables. "Scatter plot" is one of the way to get the idea about whether two variables are related or not. Theoretically, we can find out the association between two variables by calculating coefficient of correlation. It is denoted by r and value ranges from -1 to 1 i.e



If r = -1, perfect negative correlation

If r = +1, perfect positive correlation

If r > 0, positive correlation (when two variables are directly related)

If r < 0, negative correlation (when two variables are inversely related)

If r = 0, no correlation

Excel formula for Correlation
=CORREL(Variable 1 range, Variables 2 range)

Let given bivariate data (say variable x and y) is $x_1, x_2, x_3, \dots\dots\dots, x_n$ and $y_1, y_2, y_3, \dots\dots\dots, y_n$ then

Methods of computing correlation coefficient

1. **Karl Pearson's coefficient of correlation / Product moment of coefficient of correlation :**

   (i) **Direct method for ungrouped data (when deviations are taken from mean):**

$$r = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2 . \sum(y-\bar{y})^2}}$$

   (ii) **Direct method for grouped data:**

$$r = \frac{N\sum XY - \sum X . \sum Y}{\sqrt{N\sum X^2 - (\sum X)^2} . \sqrt{N\sum Y^2 - (\sum Y)^2}}$$

   (iii) **When deviations are taken from assumed mean (Assumed mean method):**

$$r = \frac{N\sum d_x d_y - \sum d_x . \sum d_y}{\sqrt{N\sum d_x{}^2 - (\sum d_x)^2} . \sqrt{N\sum d_y{}^2 - (\sum d_y)^2}}$$

Example 44: Result of two simultaneous tests conducted in a class have been given in following table. Calculate the Karl Pearson's coefficient of correlation for scores obtained by students in these two different tests.

Name of student	Priya	Rahul	Tanya	Priyanka	Sneha	Naveen	Neeraj	Sunil
Score in first test	1	3	4	6	8	9	11	14
Score in second test	1	2	4	4	5	7	8	9

Solution :

Name	**Score in first test (X_i)**	**Score in second test (Y_i)**	$(x-\bar{x})$	$(x-\bar{x})^2$	$(y-\bar{y})$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
Priya	1	1	-6	36	-4	16	24
Rahul	3	2	-4	16	-3	9	12
Tanya	4	4	-3	9	-1	1	3
Priyanka	6	4	-1	1	-1	1	1
Sneha	8	5	1	1	0	0	0
Naveen	9	7	2	4	2	4	4
Neeraj	11	8	4	16	3	9	12
Sunil	14	9	7	49	4	16	28
	56	40		132		56	84

$$\bar{x} = \frac{56}{8} = 7 \; and \; \bar{y} = \frac{40}{8} = 5$$

We know that Karl Pearson's coefficient of correlation $r = \dfrac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2 . \sum(y-\bar{y})^2}}$

$$r = \frac{84}{\sqrt{132 \times 56}} = 0.977$$

Hence there is a very high correlation between the given bivariate data.

Excel formula for Karl Pearson coefficient of Correlation
=PEARSON(Variable 1 range, Variables 2 range)

2. Spearman's rank correlation

Spearman rank correlation analysis can be used instead of Pearson's correlation coefficient when either given data is not normally distributed or there exist outliers. This method is not sensitive to outliers as it uses ranks for calculations, so value of observations does not affect.Spearman correlation coefficient is represented by r^s

In this method, the x and y variables are ranked and the ranks of x are compared to the ranks of y:

Steps of calculating coefficient of Spearman's correlation

Step 1: Put ranking to each value of both given variables (x, y) in descending order

Step 2: Find rank difference (d_i = rank x_i - rank y_i)of corresponding observations. If ranks of two items are same, then take their average.

Step 3: Square the rank difference

Step 4: Use formula given below to calculate the coefficient of Spearman's correlation

Case 1: If there are no tied ranks, then:

Spearman's rank correlation coefficient $r_s = 1 - \frac{6\sum d_i^2}{n(n^2-1)}$

where d_i = rank x_i – rank y_i

n = number of observations

Case 2: If rank associated with two or more observations are same, then:

Spearman's rank correlation coefficient $r_s = 1 - \frac{6\left(\sum d_i^2 + \frac{\sum m(m^2-1)}{12}\right)}{n(n^2-1)}$

where d_i = rank x_i – rank y_i

n = number of observations

m = Number of times a particular rank has repeated itself

Example 45: Calculate the Spearman's rank correlation for the below given data;

IQ	99	120	98	102	123	105	85	110	117	90
EQ	3	0	30	45	16	25	17	24	26	5

Solution :

IQ (X_i)	EQ (Y_i)	Rank (X_i)	Rank (Y_i)	Rank diff (d_i)	$(d_i)^2$
99	3	4	2	2	4
120	0	9	1	8	64
98	30	3	9	6	36
102	45	5	10	5	25
123	16	10	4	6	36
105	25	6	7	1	1
85	17	1	5	4	16
110	24	7	6	1	1
117	26	8	8	0	0
90	5	2	3	1	1
					184

We know that Spearman's rank correlation coefficient $r_s = 1 - \frac{6\sum d_i^2}{n\left(n^2-1\right)}$

$$r_s = 1 - \frac{6(184)}{10\left(10^2-1\right)} = -\ 0.115$$

As the coefficient of above data is negative, we can say that there isn't much of a correlation between these two variables.

Excel formula for Spearman's rank correlation
=CORREL(rank range of Variable 1, rank range of Variable 2)

Question 10: ICC team ranking of men's Cricket for the month of May 2020 is given in following table. Calculate correlation coefficient for ODI and test rankings.

Sr. No.	Team	Rating of test matches	Rating of ODI matches
1	Australia	116	107
2	New Zealand	115	116
3	India	114	119
4	England	105	127
5	Sri Lanka	91	85
6	South Africa	90	108
7	Pakistan	86	102
8	West Indies	79	76
9	Bangladesh	55	88
10	Zimbabwe	18	39

Source: https://www.icc-cricket.com/rankings/mens/team-rankings/test

10.12 Applications of descriptive statistics using real time data

Descriptive statistics has various real-world applications and become part of business's arsenal now a day. It helps in planning strategies and making informed decisions. Some of the commonly used applications are summarized as

1. Conducting census, surveys, research and interpreting their scores widely uses various statistical tools. http://censusindia.gov.in/

2. Forecasting currency exchange rates, understanding the pattern of change in value of a particular currencies wr.t. some other currencies based upon the purchase power parity and further parameters.

   https://www.rbi.org.in/scripts/ReferenceRateArchive.aspx

3. Central Pollution Control Board (CPCB) advises the Govt. and citizens on matters concerning prevention and control of water, air pollution and improvement of the quality of air based upon the collection of water and air data and their statistical analysis. https://www.cpcb.nic.in/

4. Predicting natural calamities and weather based up the data sent by satellites and various observatories. https://mausam.imd.gov.in/

5. NavIC (Indian Regional Navigation Satellite System (IRNSS)) provides accurate position information service to users in India as well as some other extending regions based upon the data sent by constellation of 7 satellite https://www.isro.gov.in/irnss-programme

6. Big Data Analytic Companies (McKinsey, IBM, Deloitte, etc.) use it for consultation, based upon the statistical analysis of existing data.

7. For running marketing campaigns based upon the tabulation of past events such as localized sales, customer interests, arrangement of social metrics from Tweets, Facebook likes and followers.

8. Ranging from planning of national level policies / programs / schemes to pilot testing is always based upon the statistical analysis of financial market (Growth, Trends, Assumptions) and data previous data.

Online Resources

1. Statistics course on National Repository of Open Educational Resources Course:https://nroer.gov.in/home/topic_details/55b1f73181fccb7926fe54e2?nav_li=55b1f72181fccb7926fe5451,55b1f73081fccb7926fe54d5,55b1f73181fccb7926fe54e2

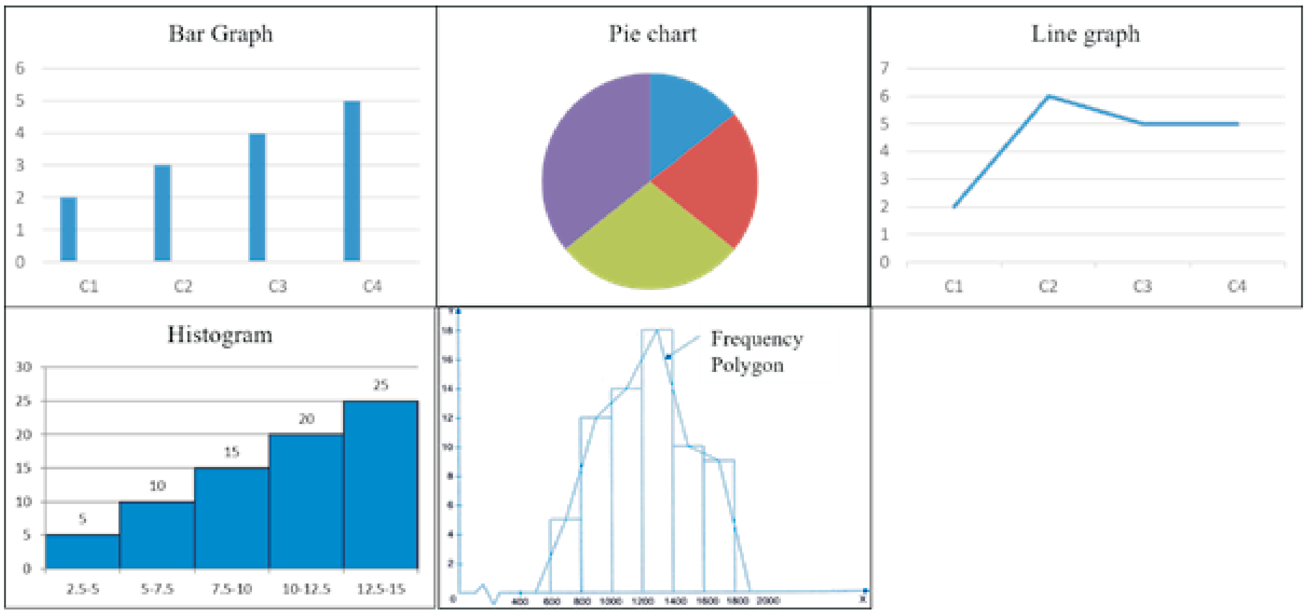
2. YouTube channel
http://www.zstatistics.com/videos/#/descriptive-statistics

3. Khan Academy Course:
https://www.khanacademy.org/math/in-in-grade-11-ncert/in-in-statistics-2

4. http://onlinestatbook.com/

Unit Summary

Representation of data:



Measures of central tendency (Mean is the most common measure of central location)		
Measure	**Definition**	**Symbol**
Mean	Sum of values, divided by total number of values	$\overline{X}$
Median	Middle point in data set that has been ordered	MD
Mode	Most frequent data value	None

Formulae			
	Individual Series	**Discrete Series**	**Continuous Series**
Measures of central tendency			
Mean Direct method	$\frac{\text{Sum of all observations}}{\text{Number of observations}}$		$\frac{\sum f_i x_i}{N}$
Assumed mean method		$a+\frac{\sum f_i d_i}{\sum f_i}$	$a+\frac{\sum f_i d_i}{\sum f_i}$
Step deviation method		$a+\frac{\sum f_i u_i}{\sum f_i}\times h$	$a+\frac{\sum f_i u_i}{\sum f_i}\times h$
Median	$\left(\frac{n+1}{2}\right)^{th}$ item, ifodd observations Average of $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ items, if even observations		$l+\left(\frac{\frac{n}{2}-cf}{f}\right)\times h$ where size of $\left(\frac{n}{2}\right)^{th}$ item determines the median class.
Mode	Most frequent item	Most frequent item	$l+\left(\frac{f_1-f_0}{2f_1-f_0-f_2}\right)\times h$
Measures of dispersion			
Range			
Quartile Deviation	QD = $\frac{Q_3-Q_1}{2}$, Where Q1 =$\frac{1}{4}(n+1)^{th}$ value Q3 =$\frac{3}{4}(n+1)^{th}$ value	QD = $\frac{Q_3-Q_1}{2}$, Where Q1 = $\frac{1}{4}(n+1)^{th}$ value Q3 = $\frac{3}{4}(n+1)^{th}$ value	QD = $\frac{Q_3-Q_1}{2}$, Where Q1 is the size of $\left(\frac{n}{4}\right)^{th}$ value Q_3 is the size of $\left(\frac{3n}{4}\right)^{th}$ value

			$Q_1 = l + \left(\dfrac{\frac{n}{4} - cf}{f}\right) \times h$ $Q_2 = l + \left(\dfrac{\frac{3n}{4} - cf}{f}\right) \times h$
Mean Deviation MD about mean MD about median	$\frac{1}{N}\sum_{i=1}^{n} \lvert x_i - \bar{x} \rvert$ where $\bar{x}$= mean $\frac{1}{N}\sum_{i=1}^{n} \lvert x_i - M \rvert$ where M = median	$\frac{1}{N}\sum_{i=1}^{n} f_i \lvert x_i - \bar{x} \rvert$ mean $\frac{1}{N}\sum_{i=1}^{n} f_i \lvert x_i - M \rvert$ where M = median	$\frac{1}{N}\sum_{i=1}^{n} f_i \lvert x_i - \bar{x} \rvert$ where $\bar{x}$= mean $l + \left(\dfrac{\frac{n}{2} - cf}{f}\right) \times h$
SD	$(\sigma) = \sqrt{\frac{1}{N}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}$	$(\sigma) = \sqrt{\frac{1}{N}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}$	$(\sigma) = \sqrt{\frac{1}{N}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}$
	Short cut method of S.D. $(\sigma) = \frac{h}{N}\sqrt{N\sum_{i=1}^{n} f_i y_i^2 - \left(\sum_{i=1}^{n} f_i y_i\right)^2}$ Note: S.D. is the most common descriptive measure of spread		
	Measures of position		
Percentile Rank	$PR = 100 - \left(\dfrac{100R - 50}{N}\right)$		$PR = \dfrac{100}{N} + \left(cf + \dfrac{x}{h} \times f\right)$
Quartile Rank	Q1 = $\frac{1}{4}(n+1)^{th}$ Q3 = $\frac{3}{4}(n+1)^{th}$		$Q_1 = l + \left(\dfrac{\frac{n}{4} - cf}{f}\right) \times h$ $Q_3 = l + \left(\dfrac{\frac{3n}{4} - cf}{f}\right) \times h$

- **Skewness denotes the degree of asymmetry in the data**
  - **Positive skewness:** Distribution has long tail on right and **Mean > Median > Mode**
  - **Negative Skewness:** Distribution has long tail on left and **Mean < Median < Mode**
  - **Zero Skewness:** When skewness is zero

Measure of skewness: Coefficient of skewness
1. Pearson's 1st coefficient of skewness $= \dfrac{Mean - Mode}{S.D.}$
2. Pearson's 2nd coefficient of skewness $= \dfrac{3(Mean - Median)}{S.D.}$
3. Quartile coefficient of skewness $= \dfrac{Q_3 + Q_1 - 2Q_2}{S.D.}$
4. Percentile coefficient of skewness $= \dfrac{P_{90} + P_{10} - 2Median}{P_{90} - P_{10}}$
5. Decile coefficient of skewness $= \dfrac{D_9 + D_1 - 2Median}{D_9 - D_1}$

- **Kurtosis : Degree of 'peakedness' of frequency curve.**
  - **Positive Kurtosis:** A frequency distribution having a **sharp peak**
  - **Negative Kurtosis:** A frequency distribution having a **blunt peak**
  - **Zero Kurtosis:** A frequency distribution having a **moderate peak.**

- **Karl Pearson's coefficient of correlation:**

Ungrouped data: $r = \dfrac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2 . \sum(y-\bar{y})^2}}$

Direct method for grouped data: $r = \dfrac{N\sum XY - \sum X.\sum Y}{\sqrt{N\sum X^2 - (\sum X)^2} . \sqrt{N\sum Y^2 - (\sum Y)^2}}$

Assumed mean method: $r = \dfrac{N\sum d_x d_y - \sum d_x . \sum d_y}{\sqrt{N\sum d_x^2 - (\sum d_x)^2} . \sqrt{N\sum d_y^2 - (\sum d_y)^2}}$

- **Spearman's rank correlation coefficient**

When there are no tied ranks $r_s = 1 - \frac{6\sum d_i^2}{n\left(n^2-1\right)}$

When rank of two or more observations are same $r_s = 1 - \frac{6\left(\sum d_i^2 + \frac{\sum m\left(m^2-1\right)}{12}\right)}{n\left(n^2-1\right)}$

Further Exploration

- **Additional resources**

  Collect data from National Crime Record Bureau (NCRB) website https://ncrb.gov.in/ and graphically plot the various data in order to visualize the given data.

- How to calculate all the descriptive statistical results of given data in one go:

Step 1: Click the **File tab,** click Options, Click **Add-ins** category

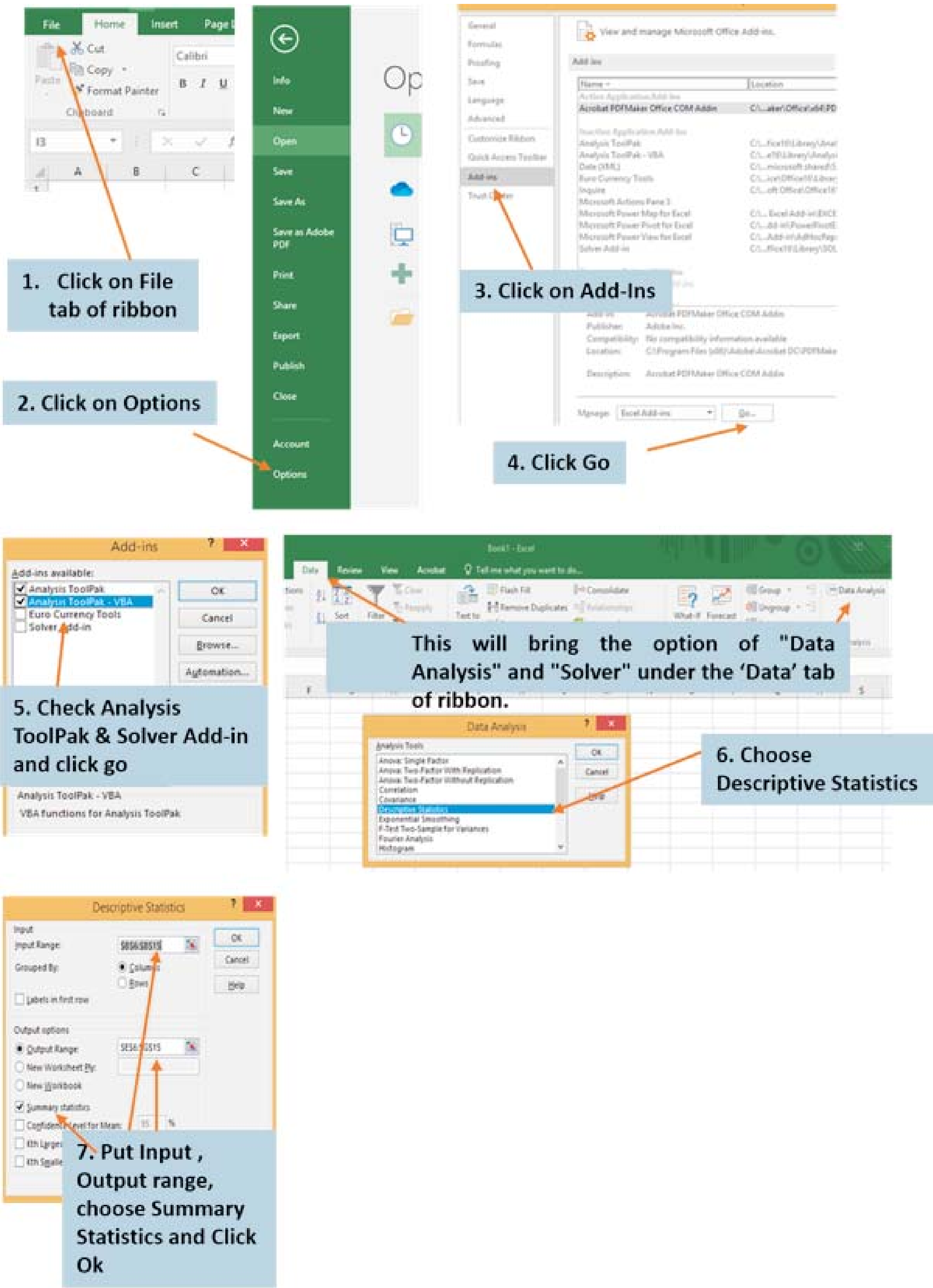
Step 2: In **Manage** box, select **Excel Add-ins** and click **Go**

Step 3: In the Add-Ins box, check the Analysis **Tool Pak & Solver Add-in** checkbox, and then click **OK**

This will bring the option of "Data Analysis" and "Solver" under the 'Data' tab of ribbon.

Step 4: Select the given data and output range with following steps

Click **Data** tab, Click **Data Analysis,** Click **Descriptive Statistics,** Select desired **Input and Output Range,** Choose **Summary Statistics** and then click **OK**


1. Click on File tab of ribbon
2. Click on Options
3. Click on Add-Ins
4. Click Go
5. Check Analysis ToolPak & Solver Add-in and click go
This will bring the option of "Data Analysis" and "Solver" under the 'Data' tab of ribbon.
6. Choose Descriptive Statistics
7. Put Input , Output range, choose Summary Statistics and Click Ok

केन्द्रीय माध्यमिक शिक्षा बोर्ड
भारत
असतो मा सद्गमय

Chapter XI

Basics of Financial Mathematics

Learning Objectives

After completion of this unit the students will be able to

- explain the origin and history of interest rate
- learn various forms of interest rate
- comprehend practical applications of interest rate in real life situation.
- have an outlook of various economic theory associated with interest rate
- Explain the relevance of financial mathematics in business and personal life.
- Develop understanding on the concepts associated with financial mathematics.
- Explain the steps involved in the computation of Income Tax and GST.
- Develop understanding on the practical applications of net present value; present value interest factor; future value interest factor and annuity.
- Calculate electricity, water and other utility bills.

11.1 Introduction

Finance and Mathematics are pervasive. Be it business activities, economic activities and philanthropic activities, finance and mathematics goes hand in hand. In other words, they are complementary to each other. Even in our daily life we observe that almost every activity where there is usage of money finance and mathematics is applicable.

The topic of Financial Mathematics is an integration of finance and mathematics. This topic would assist immensely in comprehending the practical applications of significant concepts like, interest rate; annuity; net present value; discounting; compounding; electricity bills; water supply bills; other utility bills etc.

After reading this unit a student would be able to understand the basis of determination of interest rates on various banking and other financial products, thereby, developing the ability to select appropriate financial products on the basis of interest rate, annuity, discounting and compounding.

Further, this unit would impart practical knowledge pertaining to reading of electricity, water and other utility bills, thereby making them conversant about the grey areas, like, overcharging of bills; incorrect reading of the meters; faults in the meters; basis of calculation of electricity, water and other utility charges.

Unit Structure

Unit VII Basics of Financial Mathematics	**Sub-topics**
Interest and interest rate	Origin of the concept of interest; Forms of interest rate; practical applications of interest rate; economic theories associated with interest rate.
Accumulation with simple and compound interest	Meaning and significance of simple and compound interest; formulae; calculations under simple and compound interest rates; compound interest rates application on various financial products etc.

	Video Link for reference: https://www.khanacademy.org/economics-finance-domain/core-finance/interest-tutorial/compound-interest-tutorial/v/introduction-to-compound-interest
Simple and compound interest rates with equivalency	Annual Equivalent Rate Youtube link: https://www.youtube.com/watch?v=1gxpwltFlnw
Effective rate of interest	Concept and practical applications, especially in calculating coupon interests on bonds / debentures, wherein interest is compounded half yearly / quarterly. Youtube link: https://www.youtube.com/watch?v=86_OAx0E6kI
Present value, net present value and future value	Concept and its applications; concept of compounding and discounting; usage of PVAF, FVAF tables; computation of net present value; application of net present value in capital budgeting decisions etc. Youtube link: https://www.youtube.com/watch?v=zGRVVSC4UUQ https://www.youtube.com/watch?v=Dtot7qLEtPc Case studies / Caselets on net present value from various reference books on Financial Management would be discussed.

Annuities, calculating value of regular annuity	Immediate Annuity; Annuity Due; Deferred Annuity; Perpetuity and General Annuity Youtube link: https://www.youtube.com/watch?v=Rq66DqfDQf8 https://www.youtube.com/watch?v=joBu9TnFngQ
Simple applications of regular annuities (up to 3 period)	-------------------do-----------------
Tax, calculation of tax and simple applications of tax calculation in Goods and service tax, Income Tax	Fundamentals of taxation; direct and indirect tax; tax incidence and impact; Income Tax- Assessment of Individuals; GST- Integrated Goods and Services (IGST); State Goods and Services Tax (SGST); Central Goods and Services Tax (CGST) and Union Territory Goods and Services Tax (UTGST). Calculation of GST in hospitality sector; power sector etc. Project: Practical questions from various reference books on Income Tax and Indirect Tax (GST) would be discussed.
Bills, tariff rates, fixed charge, surcharge, service charge	Types of bills; tariff rates- its basis of determination; concept of fixed charge; service charge and their applications in various sectors of Indian economy. Refer: https://www.dvvnl.org/UploadFiles/tariffPlan/NPCL%20Tariff%20Order%20-%20Ver%207.pdf
Calculation and interpretation of	Crucial components of electricity bill; water supply and other supply bills; overcharging of electricity and

electricity bill, water supply bill and other supply bills	water supply bills; units consumed in electricity bills; consumer protection laws for redressal of complaints relating to overcharging in electricty, water supply and other supply bills; concept of Ombudsman. Youtube link: https://www.youtube.com/watch?v=4WG8TBi50xU https://www.youtube.com/watch?v=x5lnOgkdBic Refer: https://www.theenergydetective.com/bills Project: Students would be assigned with the task of taking reading of power consumption of various electrical gadgets installed in their residence and compute the electricity bill based on the KWH concept.
(Comparing interest rates on various types of savings; calculating income tax; electricity bills, water bill; service surcharge using realistic data)	Interest rates on various savings, fixed and recurring deposits products; steps involved in computation of income tax of individuals and factors considered; computation of electricity bills based on realistic consumption of electric units of households in delhi (data may be accessed from BSES Rajdhani Power Ltd.)

11.2 INTEREST & INTEREST RATES

Before you start you should know how to:

Familiarize with different banking transactions helpful in getting a quick overview of the concepts. However, this topic is an attempt to cover everything from very basics and emphasis is given to cover any missing grounds.

In simple terms, Interest is a fee charged on the money that is borrowed.

When you borrow money from a bank, you pay interest for the use of banks money. When you deposit money in the Savings Account you are paid interest.

When you borrow money from someone or use somebody else's money

↓

You have to pay service charge to him.

↓

This amount is paid back to the lender along with the original amount borrowed.

↓

This is known as INTEREST

Origin of the Concept of interest

Simply put, Interest rate is the cost of borrowing or the price for lending. However, the meaning of interest rate kept on evolving through time and civilizations.In the ancient middle eastern civilization, the practice of "food money" was prevalent. People lent seeds and livestock to others in exchange for more amount or numbers of the same in return. The argument behind this was that seeds and livestock could reproduce and multiply in numbers. But when coins became a medium for exchange, and since coins can not multiply on its own, the concept of interest rate on the coins borrowed was introduced.

In the medieval era loans were obtained mostly during period of extremities like a bad harvest. Since under such circumstances charging interest was viewed morally wrong, religious institutions forbade interest. The argument was that loans were used for consumption and not production.

But in the modern era, with the advent of industrialization and urbanization, people borrowed money with the intention of production and starting new businesses. Thus, many financial institutions came up to provide loans at attractive interest rates.

Earning Interest: When you save money in the bank, they will pay you to let them look after your money. This is called Earning Interest.

Example- If you save ₹ 100 in an account with an interest rate of 5%, after a year you will have ₹105.

Earning Interest is a kind of income for a person on the money invested by him.

Paying Interest: If you borrow money, banks will charge you for this. This extra charge is called Paying Interest.

Example- If you borrow ₹ 100 with an Interest Rate of 15%, after a year you will owe them ₹ 115.

Paying interest is a kind of expense for a person on the money borrowed by him.

What is Interest Rate: An Interest Rate is the amount of interest due per period, as a proportion of the amount lent, deposited or borrowed (called the principal amount). So, an interest rate is the percentage of principal charged by the lender for the use of its money.

An Interest Rate is the overall amount that has to be paid back over and above the original amount borrowed.

Interest Rates can be low or high depending on situation to situation.

Impact of Interest Rates:

If Interest Rates are Low?

If the Interest Rates are lowered, this is positive for the consumers as their mortgage repayments are low. Also, lower interest rates means it is less worthwhile to save money in the banks as the interest earned is low. The higher the Interest rate, the more consumers earn by saving their money in the bank.

If Interest Rates are High?

If the Interest Rates are high, this is negative for the business because they have increased costs in relation to their loan repayments. On the other hand, higher interest rates mean it is more worthwhile to save money in the banks as the interest earned is high.

Offered Interest Rates vary from product to product and from bank to bank with no. of factors contributing to the rate of Interest.

Types of Interest Rates:

There are essentially three main types of interest rates:

Nominal Interest Rate | Effective Rate | Real Interest Rate

i) **Nominal Interest Rate -** The nominal interest of an investment or loan is simplified the stated rate on which the interest payments are calculated. Essentially, this is the rate on which savings accrue interest over a period of time.

For example: Investment of ₹ 10,000 at a nominal interest rate of 5% over 1 year would earn the investor ₹ 500.

ii) **Effective Rate -** The effective Interest Rate takes into account compounding over the full term of the investment. It is often used to compare annual rates with different compounding term (daily, monthly, annually, etc.)

iii) **Real Interest Rate -** It is useful when considering the impact of inflation on the nominal interest rate.

11.3 ACCUMULATION WITH SIMPLE AND COMPOUND INTEREST

Interest is charged using one or two methods. These are-

Simple Interest Method | Compound Interest Method

i) **SIMPLE INTEREST METHOD-** Under this method, the interest is charged only on the amount originally lent (principal amount) to the borrower. Interest is not charged on any accumulated interest under this method.

Formula for calculating Interest under Simple

Interest Method- Interest = P x i x n

Where: **P = Principal Amount**

i = Interest Rate

n = No. of periods

Example: A loan of ` 10,000 has been issued for 6 years. Compute the amount to be repaid by the borrower to the lender if simple interest is charged @ 5% per year.

Solution : P = ₹ 10,000, i = 5%, n = 6

By putting the values of P, i , n into the simple interest formula-

I = P x i x n

= 10,000 X 5% X 6

= ₹ 3,000

At the end of 6th year, the amount=Principle + Interest of ₹ 10,000 + ₹ 3,000 = ₹ 13,000 will be repaid to the lender.

ii) **COMPOUND INTEREST METHOD-** Under this method, the Interest is charged on the principal plus any accumulated interest. The amount of Interest for a period is added to the amount of principal to compute the Interest for next period. In other words, the interest is reinvested to earn more interest. The interest may be compounded monthly, quarterly, semiannually or annually.

When interest is added to the Principal amount, the resulting figure is known as Compound Amount.

Formula for calculating Compound Amount and Compound Interest.

Compound Amount Formula: $A = P(I + i)^n$

Where:

A = Compound Amount

P = Principal Amount

i = Interest Rate

n = No. of Periods

Compound Interest Formula

Compound Interest = Compound Amount (-) Principle Amount

Principal Amount

Example: The Bank has issued a loan of ₹ 1000 to a sole proprietor for a period of 5 years. The Interest Rate for this loan is 5% and the Interest is compounded annually. Compute -

(1) Compound Amount

(2) Compound Interest

Solution: Computation of Compound Amount:

$\mathbf{A = P\ (I + i)^n}$

$= 1000 \times (I + 5\%)^5$

$= 1000 \times (I + 0.05)^5$

$= 1000 \times (1.05)^5$

$= 1000 \times 1.276'$

$=$ ₹ 1276

Interest

1) Divide ₹ 43500 into two parts so that the simple interest an the fist when deposited for one year at 9% per annum and that on the second when deposited for 2 years at 10% per annum in a bank are the same.

Solution: Let the first part be x then 2nd part (43500-x)

$$SI = \frac{x.9.1}{100} = \frac{9x}{100} \text{.......(i)}$$

for 2nd part $SI = \frac{(43500 - x).10.2}{100}$ .....(ii) from question

$$\frac{9x}{100} = \frac{(43500 - x)20}{100}$$

$9x = 87000\text{-}20\,x$

$x = ₹\,30000$

Therefore, $43500 - x = ₹13500$

2) At what rate percen per annum will a sum of money tripple in 16 years

Solution : Let $P = x$

$A = 3x$

$SI = 3x - x = ₹\,2\,x$

$$R = \frac{SI \times 100}{P \times T} = \frac{2x \times 100}{x \times 16} = 12.5\%$$

3) A Sum of money lent art at simple interest amount to ₹ 2200 in one year and ₹ 2800 in 4 years. Find the sum of money and the rate of interest

Solution : A: Let $P = x$

$$\text{Amount} = P + \frac{Prt}{100} = 2200 = P(100 + r) = 2200 \times 100 \quad (\text{time} = 1\,\text{yr}) \ldots\ldots\, (i)$$

Again : Amount in 4 years

$$A = P + \frac{Pr.4}{100} = 2800 = P(100 + 4r) = 2800 \times 100 \ldots\ldots\, (ii)$$

Dividing (ii) by (i)

$$\frac{\cancel{P}(100 + 4\,r)}{\cancel{P}(100 + r)} = \frac{2800 \times 100}{2200 \times 100} = \frac{28}{22} = \frac{14}{11}$$

$1100 + 44\,r = 1400 + 14\,r$

$30\,r = 300 = r = 10\%$

From equation 1 $P(100 + 10) = 2200 \times 100$

$$P = \frac{\overset{20}{\cancel{2200}} \times 100}{\cancel{110}} = ₹\,2000$$

Computation of Compound Interest:

Once the Compound Amount has been computed, the amount of interest earned over the investment period can be computed by subtracting Principal amount from Compound Amount.

In this example, the Principal Amount is ₹ 1000 and the Compound Amount as computed above is ₹ 1276. The amount of Compound Interest for the 5 years period can be computed as follows:

Compound Interest = Compound Amount (-) Principle Amount

= 1276 - 1000

= ₹ 276

11.4 SIMPLE AND COMPOUND INTEREST RATES WITH EQUIVALENCY

❖ Compound Interest is the addition of interest to the principal sum of a loan or deposit or in other words interest on interest. It is the result of reinvesting interest, rather than paying out, so that the interest in the next period is then earned on the principal sum plus the previously accumulated interest. Compound Interest is standard in finance and economics.

❖ Compound Interest is contrasted with simple interest, where previously accumulated interest is not added to the principal amount of the current period, so there is no compounding. The simple annual interest rate is the Interest amount per period, multiplied by the number of periods per year. The simple annual interest rate is also known as the Nominal interest Rate.

(not to be confused with the interest rate not adjusted for inflation, which goes by the same name).

❖ **Concept of Equivalency -** The nominal rate cannot be directly compared between loans with different compounding frequencies. Both the nominal interest rate and the compounding frequency are required in order to compare interest- bearing financial instruments.

To help the consumers compare retail financial products more fairly and easily, many countries require financial institutions to disclose the annual compound interest rate on deposits or advances on a comparable basis.

The Interest Rate on an annual equivalent basis may be referred to variously in different markets as Annual Percentage Rate (APR), Annual Equivalent Rate (AER), effective interest rate, effective annual rate, annual percentage yield and other terms. The effective annual rate is the total accumulated interest that would be payable up to the end of one year, divided by the principal sum.

- There are usually two aspects to the rules defining these rates:-

  1) The rate is the annualized compound interest rateand

  2) There may be charges other than interest. The effect of fees or taxes which the customer is charged and which are directly related to the product, may be included. Exactly which fees and taxes are included or excluded varies by country may or may not be comparable between jurisdictions because the use of such terms may be inconsistent, vary according to local practices.

Compound Interest

1. Find the amount of Rs 20,000 after 10 years at 8% converted quarterly for the first 4 year and at 6% compounded monthly thereafter.

Ans. : Case - 1

For the first 4 years

n1 $= 4 \text{x} 4 = 16$years

$$\text{rate} = i_1 = \frac{8}{4 \times 100} = 0.02$$

Case - 2 for the next 6 years

$$n_2 = 6 \times 12 = 72$$

$$i_2 = \frac{6}{12\text{x}100} = 0.005$$

Amount on compound interest CI after 10 years

$$
\begin{aligned}
&= P(1+i_1)^{n_1}\ (1+i_2)^{n_2} \\
&= 20000(1+0.02)^{16}\ (1+0.005)^{72} \\
&= 20000 \times (1+0.02)^{16} \times (1.005)^{72} \\
&= 20000 \times 1.3727 \times 1.4320 \\
&= \quad 39314.128 \quad = \quad 39314.13
\end{aligned}
$$

2) The difference between the CI and SI on a centrain sum of money at 40% per annum for 2 years is Rs 500. Find the sum

Ans. Let principal P= 9

$$
\begin{aligned}
SI &= \frac{P \times 10 \times 2}{100} = \frac{P}{5} \\
CI &= P\left[\left(1+\frac{10}{100}\right)^2 - 1\right] = P\left(\frac{121-100}{100}\right) = \frac{21}{100}p
\end{aligned}
$$

$$
\because \quad \frac{21}{100}P - \frac{P}{5} = 500
$$

$$
\Rightarrow \quad 21P - 20P = 500 \times 100
$$

$$
\Rightarrow \quad P = ₹\,50000
$$

Example:

(a) ₹1000 is deposited into savings account paying 20% per annum. Compounded annually. At the end of 1 year Compound interest 1000 X 20% = ₹ 200

Interest rate is credited to the account Amount = 1000+200 = ₹ 1200. The account then earns Compound interest of 1200 X 20% = ₹ 240 in the second year.

(b) A rate of 1% per month is equivalent to a simple annual interest rate (nominal rate) of 12%but allowing for the effect of compounding the annual equivalent compound rate is 12.68% per annum $\{(1.01)^{12} - 1\}$

(c) The Interest on Corporate Bonds and Government Bonds is usually payable twice yearly. The amount of interest paid (each six months) is the disclosed interest rate divided by two and multiplied by the principal. The yearly compounded rate is higher than the disclosed rate.

- It is sometimes mathematically simpler to use continous compounding which is the limit as the compounding period approaches zero.
- Continuous compounding can be thought of as making the compounding period infinitely small, which is achieved by taking the limit as n goes to infinity.

Practical Application of interest rate:

Let us try to understand application of interest rates from a bank's perspective. A bank is a financial institution which engages in borrowing and lending with public and private entities. Bank charges interest on lending and provides interest upon borrowing. The fine balance between the two interest rates drives the character of the economy. Borrowers come to bank for loans for businesses, education, personal and various other purposes. The bank agrees to do so but at a certain fixed rate. The bank may not always have all the money by itself to lend. So, it offers an attractive interest to entities agreeing to deposit their money with the bank. So, from a bank's perspective higher positive interest rate is the result of lack of sufficient loanable amount. This encourages entities to deposit and save more. A situation when a bank is getting overwhelmed with deposits of fund, the interest comes down encouraging entities to take loans and spend more. In certain rare situations, like aeconomic recession, the rate may approach the limiting value of zero discouraging savings and encouraging more and more spending. It is interesting to note that in recent times many countries like Sweden, Switzerland, and Japan have implemented negative interest rate making savings an unattractive option. This tends to be the case when people hoard more and more money without spending it.

Clearly, interest rate is a very important tool for the government, financial and regulatory institutions to control and regulate the character of the economy.

Economic Theory associated with interest rate:

The great Greek philosopher Aristotle's view had a great influence on the society. He condemned the charging of interest on lending money as "unnatural". He believed that the nature of money is to facilitate exchange and was strongly against making gain from money itself.

David Hume's classic essay "of money" had an important proposition on interest and its variations. He argued that interest depends on demand for borrowing, riches to supply demand and profits arising from commerce.

The classical theory of interest, which had contributions from a number of economists, argued that interest rate ensures equilibrium between demand for borrowing and lending.

Henceforth, many more sophisticated theories of economics were proposed which made great contributions to the development and understanding of the concept of interest. However, discussion of those would be beyond the scope of this chapter.

11.5 EFFECTIVE RATE OF INTEREST

The effective rate of interest is the real return on a savings account or any interest paying investment when the effects of compounding over time are taken into consideration.

It reveals the real percentage rate owed in the interest on a loan, a credit card or any debt.

It is also known as - the Effective Rate or the Annual Equivalent Rate.

- **Concept of Effective Interest Rate -** A bank certificate of deposit, a savings account or a loan offer may be advertised with its nominal interest rate as well as its effective annual interest rate.

  The nominal interest rate does not reflect the effects of compounding interest that comes up with these financial products. The effective annual interest rate is the real return.

KEY POINT: The more frequent the compounding periods, the greater the return.

Formula for Interest Rate which is Effective:

Effective Annual Interest Rate = $(1 + i/n)^n - 1$

Where: i= Nominal Interest Rate

n = Number of Periods

Example of Effective Annual Interest Rate:

Consider two offers:

- Investment A pays 10 % interest, compounded monthly.
- Investment B pays 10.1 % compounded semi-annually.

Which is better offer?

Solution: In both the cases the advertised interest rate is the nominal interest rate. The effective annual interest rate is calculated by adjusting the nominal interest rate for the number of compounding periods the financial product will experience in a period of time. In this case, that period is one year.

Following are the calculations:-

Effective Annual Interest Rate for A & B

Investment A = $(1 + 10\% / 12)^{12} - 1 = 10.47\%$

Investment B = $(1 + 10.1\% / 2)^{2} - 1 = 10.36\%$

CONCLUSION: Investment B has a higher stated nominal interest rate, but the effective annual interest rate is lower than the effective rate for Investment A. This is because Investment B compounds fewer times over the course of the year.

- **Concept of More Frequent Compounding Equals Higher Returns -** As the number of compounding periods increases, so does the effective annual interest rate. Quarterly compounding produces higher returns than semi-annual compounding, monthly compounding more than quarterly, and daily compounding more than monthly.

Below is the breakdown of the results of these different compound periods with a 10% nominal interest rate-

Semi-annually	=	10.250%
Quarterly	=	10.381%
Monthly	=	10.471%
Daily	=	10.516%

The Limits of Compounding- There is a ceiling to the compounding phenomenon. Even if compounding occurs an infinite amount of times - not just every second or microsecond but continuously - the limit of compounding is reached.

Short trick for calculating Effective Interest Rate(to be solved on Calculator)

Base = 1 year

- R/ 100 + 1 X 100 = -100 = Effective Rate (Yearly)
  → 1 time
- R/ 200 + 1 X 100 = -100 = Effective Rate (Half Yearly)
  → 2 times
- R/ 400 + 1 X 100 = -100 = Effective Rate (Quarterly)
  → 4 times
- R/ 1200 + 1 X 100 = -100 = Effective Rate (Monthly)
  → 12 times

11.6 PRESENT VALUE, NET PRESENT VALUE AND FUTURE VALUE

- **Present Value:** The concept of present value is one of the most fundamental and pervasive in the world of finance. It is the basis for stock pricing, bond pricing, banking, insurance, etc.

In simpler terms, present value accounts for time value of money.

Present Value describes how much a future sum of money is worth today. It accounts for the fact that money we receive today can be invested today to earn a return.

Formula for Present Value

$$PV = \frac{CF}{(1+r)^n}$$

Where:

CF = Cash Flow in Future Period

r = Periodic Rate of return or Interest (also called the discount rate or the required rate of return)

n = no. of periods

- **Present Value explained through an Example:** Assume that you would like to put money in an account today to make sure you have enough money in 10 years to buy a car. If you would like to have ₹10,000 in 10 years and you know you can get 5% interest per year from a savings account during that time, howmuch should you put in the account now?

Applying the Present Value formula-

CF = ₹10,000

r = 5%

n = 10

$$\text{So, } PV = \frac{CF}{(1+r)^n}$$

$$= \frac{10,000}{(1+0.05)^{10}}$$

= ₹6139.13

Interpretation: Thus, ₹ 6139.13 will give you ₹10,000 in 10 years if you can earn 5% each year. In other words, the present value of ₹10,000 is ₹6,139.13

- **Net Present Value (NPV) :** Net Present value is the difference between the present value of cash inflows and the present value of cash outflow over a period of time.

Net Present Value is used to analyse the profitability of a project.

Formula for calculating NPV

$$NPV = \sum_{n=1}^{N} \frac{CFn}{(1+i)^n} - \text{Initial Cash Investment}$$

Where:

n = Cash Flow Period

i = Interest Rate Assumption

What is Positive NPV ???

A positive NPV indicates that the projected earnings generated by a project or investment exceeds the anticipated costs.

It is assumed that an investment with a positive NPV will be profitable and an investment with a negative NPV will result in Net loss. This concept is the basis for the Net Present Value Rule, which dictates that only investments with positive NPV values should be considered.

NPV's can be calculated by using tables, spreadsheets etc.

Illustration:

Let us say X Ltd. wants to expand its business and so it is willing to invest ₹10,00,000

The investment will bring an inflow of ₹1,00,000 in first year, ₹2,50,000 in the second year, ₹3,50,000 in the third year, ₹2,65,000 in the fourth year and ₹4,15,000 in the fifth year. Assume the discount rate to be 9%.

Let us calculate NPV using the formula

Year	Flow	Present Value	Computation
0	10,00,000	10,00,000	————————
1	1,00,000	+91743	100000 / (1.09)
2	2,50,000	+2,10,419	250000 / $(1.09)^2$
3	3,50,000	+ 270264	350000 / $(1.09)^3$
4	2,65,000	+187732	265000 / $(1.09)^4$
5	4,15,000	+269721	415000 / $(1.09)^5$
		+29881	

Hence NPV is ₹29,881

Interpretation:

Since the NPV is positive, the investment is profitable and hence X Ltd. Can go ahead with the expansion.

▲ Future Value

Future Value is the value of an asset at a specific date. It measures the nominal future sum of money that a given sum of money is 'worth' at a specified time in the future assuming a certain interest rate, that is, the rate of return. It is the present value multiplied by the accumulation function.

The value does not include corrections for inflation or other factors that affect the true value of money in future.

Understanding Future Value : The future value calculation allows the investors to predict, with varying degree of accuracy, the amount of profit that can be generated by different investments. Investors are able to reasonably assume an investment's profit using the future value calculation.

Determining the future value (FV) of a market investment can be challenging because of market's volatility.

Types of Future Value:

- Future value using simple Annual Interest-

It assumes a constant rate of growth and a single upfront payment left untouched for the duration of investment.

If an investment earns simple interest, then the future value (FV) Formula is -

FV = I x (1 +(R X T))

Where:

I = Investment Amount

R=Interest Rate

T=No. of years

Example: Assume ₹1000 investment is held for 5 years in savings account with 10% simple interest paid annually. In this case the FV is ₹1000 (1 +(0.10 X 5)) = ₹1500

- Future value using Compounded Annual Interest-

With compounded interest, the rate is applied to each period's cumulative account balance. The formula for calculating the Future value of investment earning Compound Interest is

$FV = I X (1 + R)^T$

Where:

I = Investment Amount

R = Interest Rate

T = No. of Years

Example: Using the above example, the same ₹1000 for 5 years with 10% compounding interest rate would have future value of ₹1000 x $((1+0.10)^5)$ = ₹ 1610.51

11.7 ANNUITIES, CALCULATING VALUE OF REGULAR ANNUITY

An annuity is a series of payments made at equal intervals.

Examples of annuities are regular deposits to a savings account, monthly home mortgage payments, monthly insurance payments, pension payments etc.

Annuities can be classified by the frequency of payment dates. The payments may be made weekly, monthly, quarterly, yearly or any other regular interval of time.

An annuity which provides for payments for the remainder of a person's lifetime is a life annuity.

Types of Annuity:

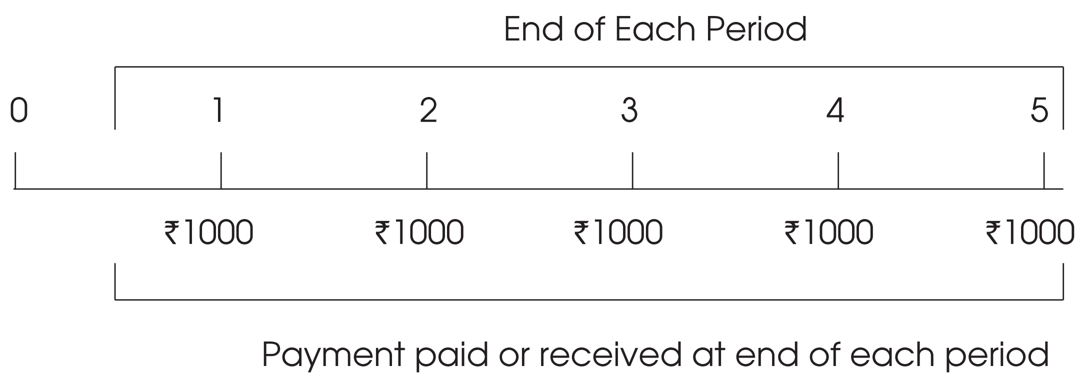
i) **Ordinary Annuity :** An ordinary annuity makes payments at the end of each period. For example - Bonds generally pay interest at the end of every six month, Example : Repayment of housing load of equal instalments.

ii) **Annuity Due :** With an annuity due, by contrast, payments come at the beginning of each period. For example - Rent, which the landlords typically require at the beginning of each month is a common example a) LIC Payment c) Recurring deposit Payment.

iii) **Deferred Annuity :** An annuity which is payable after the tapse of a number of periods is called deferred annuity. Example : Pension Plan of LIC, ordinary annuity is also known as Annuity immediate.

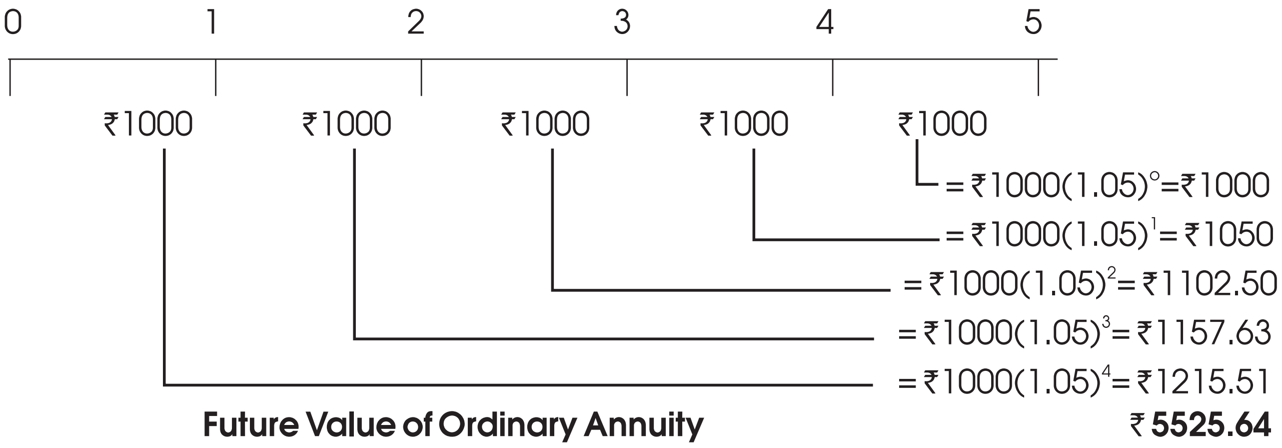
If a person plan to invest a certain amount each month or year, it will tell how much will it have accumulated as of future date.

For Example: A series of five ₹ 1000 payments made at regular intervals-



Let us assume that you invest ₹1000 every year for next 5 years at 5% Interest.

This is how much you have at the end of five year period -



Characteristics of Annuity

a) Same amounts is paid at equal time interval

b) Instalments are due at the end of each period or at the beginning of each period.

Formula for Future Value

FV $= C \times \left[\frac{(1+i)^n - 1}{i}\right]$

Ordinary Annuity

Where: C = Cash Flow per period

i = Interest rate

n = Number of Payments

In above example-

Annuity $=$ ₹1000 $\times \left[\frac{(1+0.05)^5 - 1}{0.05}\right]$

$=$ ₹1000 x 5.53

$=$ ₹ 5,525.63 or 5525.64 (roundoff)

Formula for Present Value

PV $= C \times \left[\frac{(1-(1+i)^{-n}}{i}\right]$

Ordinary Annuity

In above example-

PV

$$\text{Ordinary Annuity} = ₹\,1000 \times \left[\frac{(1-(1+0.05)^{-5}}{0.05}\right]$$

$$= ₹1000 \text{ x } 4.33$$

$$= ₹4329.49$$

Example 1 : A man deposits in a bank Rs 16000 at the end of each year, for 8 years, If the rate of interest is 10% per annum compounded annually. What would be the sum standing to his credit at the end of that period.

Ans: $C = 16000,\ i = \frac{r}{100} = \frac{10}{100} = 0.1$

$n = 8$ years

$$\text{Amount} = \frac{c}{i}\left((1+i)^n - 1\right)$$

$$= \frac{160000}{001}\left((1+0.1)^8 - 1\right)$$

$$= 160000\ \left((101)^8 - 1\right)$$

Let $x = (1.1)^8$

$\log\ x = 8 \log 1.1$ (taking log both sides)

$= 8 \times 0.0414 = 0.3312$

$x = \text{anti}\log\ 0.3312 = 2.144$

$\therefore A = 16000(2.144 - 1) = 16000 \times 1.144 = \text{Rs.}18304$

Example 2 : Find the least number of years for which an ordinary annuity of Rs 1500 per annum must run in order that its amount just exceeds Rs 30000 at 9% compound annually

Ans.: A= Rs 30000, a= Rs. 1500

$$i = \frac{9}{100} = 0.09$$

we have

$$A = \frac{a}{i}\left((1+i)^n - 1\right)$$

$$3000 = \frac{1500}{0.09}\left((1+0.09)^n - 1\right)$$

$$\frac{30000 \times 0.09}{1500 \times 100} = (1+0.09)^n - 1$$

$$1+1.8=(1+0.09)^n$$
$$2.8=(1.09)^n$$
$$\text{Log } 2.8 = n \text{ Log} 1.09$$
$$0.4472 = n \times 0.0374$$
$$n=\frac{0.4472}{0.0374}=11.957 \cong 12 \text{ years}$$

Example 3 : A firm anticipates a Capital expenditure of Rs 100000 for a new equipment in 5 years. how much should be deposited quarterly in a sinking fund carrying 12% per annum compounded quarterly to provide for purchase.

Solution : Let Rs x per quarter be deposited

Instalment = Rs. x, Time = 5 years = 20 quarters

Rate = 12% p.a. = 30% per quarter

Amount = Rs 100000

$$\text{We know} \quad A=\frac{a\left((1+1)^n-1\right)}{i}$$
$$100000=\frac{x\left((1+\frac{3}{100})^{20}-1\right)}{0.03}$$
$$x=\frac{100000\times 0.03}{(1.03)^{20}-1}$$
$$\text{Let } \ p=(1.03)^{20}$$
$$\text{Log } p = 20 \text{ Log } 1.03$$
$$=20\times 0.0128 = 0.256$$
$$p=\text{antilog } 0.256 = 1.803$$
$$x=\frac{100000\times 0.03}{1.803-1}=\frac{100000\times 0.03}{803}=\frac{3000000}{803}$$

$$\text{Log } x = \text{Log } \frac{3000000}{803} = \text{Log } 3000000 - \text{Log } 803$$
$$=6.4771-2.9047$$
$$=3.5724$$
$$c=\text{Antilog } 3.5724 = \text{Rs. } 3736$$

Example 4 : Nidhi buys a LCD set worth Rs 48000 on instalment plan under which Rs 8000 is to be paid immediately and the balance in 15 equal instalment (annualy) with 18% per annum compound interest. How much has he to pay annually

Ans. : Cost of TV Set = Rs 48000

Down payment = Rs 8000

Balance = Rs 40000

Let each instalment = x

$$P = \frac{a(1-(1+i)^{-n}}{i}$$

$$40000 = \frac{x}{0.18}\left[1-(1+0.18)^{-15}\right]$$

$$= \frac{x}{0.18}\left[1-(1+18)^{-15}\right] \qquad ....(i)$$

Let $p = (1.18)^{-15}$

$\Rightarrow \text{Log } p = -15 \text{ Log } 1.18 = .15 \times 0.0719 = -1.0785$

$\Rightarrow p = \text{anti Log } (\bar{2}.9219) = 0.0834$

$$D \Rightarrow 40000 = \frac{x}{0.18}\left[1-0.08347\right]$$

$$\Rightarrow x = \frac{40000 \times 0.18}{0.91653} = \frac{72000}{9.1653}$$

$\text{Log } x = \text{Log } 72000 - \text{Log } 9.1653$

$= 4.8573 - 0.9621 = 3.8952$

$\Rightarrow xxxxxx = \text{anti Log } 3.8952 = 7856$

hence each instalment = Rs. 7856

Example 5 : Same for a child's education, a family decides to invest Rs. 3000 at the end of each 6 month period in a fund paying 8% per year compounded semi annually. Find the amount of the investment at the end of 18 years.

Solution :

P = Rs 3000

n = 18 x 2 = 36

$$I = \frac{8}{2 \times 100} = 0.04$$

Let A be the amount of annuity then

$$A = P\left[\frac{(1+i)^n - 1}{i}\right]$$

$$= 3000\left[\frac{(1+0.04)^{36} - 1}{0.04}\right] = 3000\left[\frac{(1.04)^{36} - 1}{0.04}\right]$$

Let $x = (1.04)^{36}$

$\text{Log } x = 36 \text{ Log } 1.04 = 36 \times 0.0170 = 0.612$

$x = \text{Anti Log } 0.612 = 4.093$

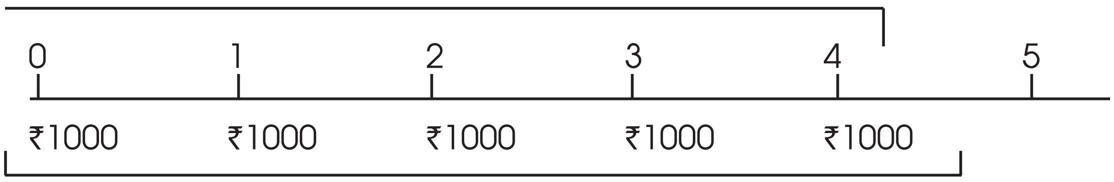
$$x = \text{Anti Log } 0.612 = 4.093$$

$$\therefore A = 3000\frac{(4.093 - 1)}{0.04} = 3000\,(77.325) = 231975$$

Amount = Rs 231975

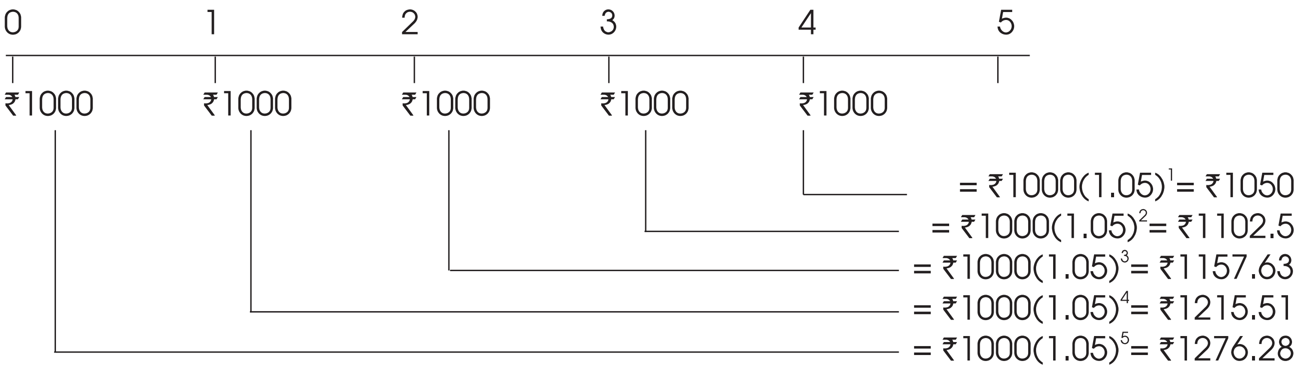
▲ Calculation of Future Value of Annuity Due

Beginning of Each Period



Payment paid or received at beginning of each period

Following the previous example-



Future Value of Annuity Due **₹ 5801.92**

Formula for Future Value

FV Annuity Due $= C\times\left[\frac{(1+i)^n-1}{i}\right](1+i)$

In above example-

FV Annuity Due = ₹ $1000\times\left[\frac{(1+0.05)^5-1}{0.05}\right](1+0.05)$

= ₹1000 x 5.53 x 1.05

= ₹ 5801.92

Formula for Present Value

PV Annuity Due $= C\times\left[\frac{(1-(1+i)^n}{i}\right](1+i)$

In above example-

PV **Annuity Due** = ₹ $1000\times\left[\frac{(1-(1+0.05)^{-5}}{0.05}\right](1+0.05)$

= ₹1000 x 4.33 x 1.05

= ₹4545.95

11.8 SIMPLE APPLICATIONS OF REGULAR ANNUITIES (Upto 3 periods)

An annuity is a series of equal cash flows, equally distributed over time.

Examples of Annuities abound -

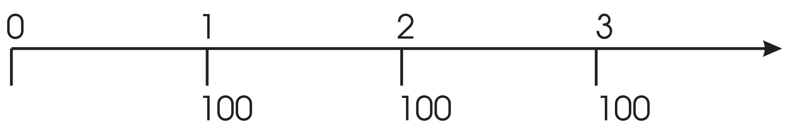
- Mortgage Payments

- Car Loan Payments
- Leases
- Rent Payment
- Insurance Payouts, and so on

If you are receiving or paying same amount of money every month (or week or year, or whatever time frame), then there is an annuity.

A regular annuity is simply an annuity where the first payment is made at the end of the period.

The picture below shows an example of a 3-period, `100 regular annuity.



Notice that we can view the annuity as a series of three Rs.100 lumpsums or we can treat the cash flows as a package.

Calculating the Future Value of a Regular Annuity

As noted, annuity can be treated as a series of lump sum cash flows.

$$FV_A = \sum_{t=1}^{N} CF_t (1+i)^{N-t}$$

Using the example shown in the time line (above) and a 9% period interest rate, we get -

$$FV_A = 100\,(1+0.09)^2 + 100\,(1+0.09) + 100$$

$$= ₹327.81$$

Analysis: Note that future value of regular annuity is in the same period as the last cash flow. So the first cash flow must be taken two periods forward, the second cash flow must be moved one period ahead and the last cash flow is already there, so it does not move at all.

Therefore, if the Interest Rate is 9%, then the future value of this annuity is ₹327.81 at the end of period 3.

The formula shown above works fine, but it is tedious if the annuity has more than a few payments. Fortunately, we can derive a closed form version of that equation which means that we don't have to iterate through a series of sums. The closed form equation is -

$$FV_A = Pmt\left[\frac{(1+i)^N - 1}{i}\right]$$

Where Pmt is the per period annuity payment amount(` 100 in our example). This formula is much easier to use, no matter how many payments there are. In this case, it gives us -

$$FV_A = 100\left[\frac{(1+0.09)^3 - 1}{0.09}\right] = 327.81$$

which is exactly the same as we got previously.

Example: Imagine that you are planning to retire in 35 years and you think you can afford to save ₹500/- per month. Further, you believe that you can reasonably earn about 8% per year without taking too much risk.

How much amount will you have accumulated at the time you retire?

In this example, Pmt is ₹500/- because you plan to save that amount each and every month. The number of periods N is 420 months and the interest rate is 0.667% per month.

In this case, we are making monthly investments so both N and i must be converted to monthly values.

Solving the closed form equation, we find that you will have –

$$FV_A = 500\left[\frac{(1+0.00667)^{420} - 1}{0.00667}\right]$$

= ₹1,146,941.24/-

Interpretation: We just found that investing ₹500 per month @ 8% per year will result in the nest egg of ₹1146941 after 35 years.

11.9 TAX, CALCULATION OF TAX AND SIMPLE APPLICATION OF TAX CALCULATION IN GOODS AND SERVICE TAX, INCOME TAX

▲ **What is Tax?**

Tax is money that people have to pay to the government.

A tax is a compulsory financial charge or some other type of levy which is imposed by a governmental organization in order to fund various public expenditures.

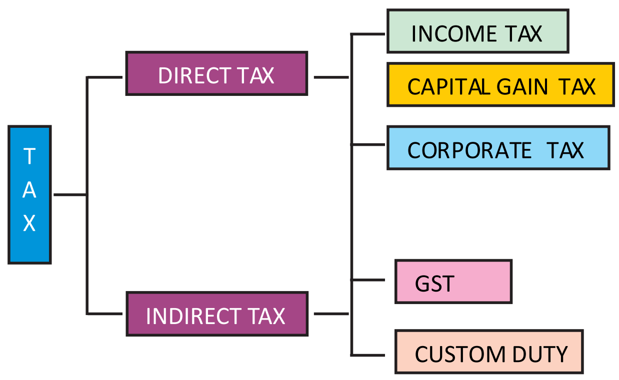
For example, taxes are used to pay for the people who work for the government, such as the military and the police, provide services such as the education and health care, and to maintain or build the things like roads, bridges etc.

So, taxes are the primary sources of income for the government through which it fulfills various projects and initiatives. Tax is vital for the country's economic progress, sustenance and development.

▲ **Taxes prevalent in India**

In India there are two types of Taxes prevailing i.e. Direct Tax and Indirect Tax. Generally, taxes which are directly collected from tax payer are called Direct Taxes and taxes which are indirectly collected are called Indirect Taxes.

As you can see in below hierarchy, Direct Tax are further divided into Income Tax, Capital Gain Tax, Corporate Tax. Similarly, Indirect Taxes are further divided into two i.e. GST (Goods and Services Tax) and Custom Duty.



Computation of Income Tax

Income from Salary	XXX
Income from House Property	XXX
Income from Business or Profession	XXX
Income from Capital Gains	XXX
Income from Other Sources	XXX
Gross Total Income	XXX
Less: Deductions (Limit `150000)	XXX
(PF, PPF, LIC, housing Loan, FD, NSC)	
Taxable Income	XXX

(Rebate of ₹12,500 is available if taxable income is upto ₹5,00,000)

Applicable Tax Rate

For the financial year 2020-21 (Assesment year 2021-2022) rate of income tax is as under

Slab	**Income Tax Rate**	**Health and Education Cess**
Upto 2.5 Lakh	Nil	4%
2.5 - 5.0 Lakh	5% of amount by which taxable income exceeds Rs 2,50,000	
5.0 - 7.5 Lakh	Rs 12500 + 10% (taxable income 2,50,000)	4%
7.5 - 10 Lakh	Rs 37500 + 15% (taxable income 7,50,000)	4%
10 - 12.5 Lakh	Rs 75000 + 20% (taxable income 10,00,000)	4%
12.5 - 15 Lakh	Rs 1,25000+25% (taxable income 12,50,000)	4%
Above 15 Lakh	Rs 187500 + 30% (taxable income 15,00,000)	4%

Example 1: Income from Salaries is ₹ 4,00,000 for the financial year 2019-20. Calculate Income Tax liability if the deductions under LIC is 50,000.

Solution -	Income from Salaries	₹ 4,00,000
	Gross Total Income	₹ 4,00,000
Less:	Deductions	₹ 50,000
	Taxable Income	₹ 3,50,000

Tax Calculation upto ₹ 2.50 lacs	NIL
(₹350000 - ₹250000) X 5%	₹ 5,000
TOTAL TAX	₹ 5,000

* Since Taxable Income (₹350000) is less than ₹500000, so Rebate of ₹12500 is available.

Hence total tax payable is NIL.

EXAMPLE 2 : Income from salaries is ₹2,00,000. Income earned from House property rentals ₹1,00,000. Income under head Business and Profession, come out to be ₹2,50,000. Income under head capital gain is ₹1,00,000.

PF contribution is ₹50,000, LIC is ₹50,000.

Calculate Income Tax Liability

SOLUTION :

Income from Salary	2,00,000
Income from house property	1,00,000
Income from Business profession	2,50,000
Income from Capital gains	1,00,000
Gross total income	6,50,000
LESS : Deductions	
PF contributions	(50,000)
L I C	(50,000)
Taxable Income	5,50,000

Calculation of Tax :

0 - 250000	NIL	
	(5,00,000 - 2,50,000) x 5%	12,500
	(5,50,000 - 5,00,000) x 20%	10,000
	Total Tax	22,500

- Since taxable Income is more than `5,00,000, no Rebate is available.

EXAMPLE 3 : Raman's monthly salary is Rs 1,93,800. HRA is 24% of Basic Salary and Transportation allowance is Rs 8424 per month. he deposits Rs 40,000 om GPF and pays Rs 45000 per month as the income tax for 11 month. How much income tax Raman still have to pay.

SOLUTION :

Salary : Rs. 1,93,800

$$HRA = \frac{1,93,800 \times 24}{100} = \text{Net } 46,512$$

$TA = \text{Rs } 8240$

Net monthly Salary = 1,93,800 + 46512 + 8424

= 2,48,736

Annual Salary = 12 x 248736 = 2,984,832

$$\text{Income tax} = 187500 + \frac{30}{100} \times (2,984,830 - 1500000)$$

$$= 6,32,949$$

$$\text{Health and education cess} = \frac{632949 \times 4}{100} = 25317.96 = 25318$$

Surcharge = 25% of Income Tax

$$= \frac{25}{100} \times 6,32,948 = 158237.25$$

$$= 158238$$

Net Income Tax = 6,32,949 + 25,318 + 1,58,237

= Rs. 816504

Tax already paid in 11 month =45000 x 11 = Rs 4,95,000

Tax Payable = - Rs 8,16,504

Rs 4,95,000

Rs. 3,21,504

Check Your Progress :

Q. Mr X had income from salary of ₹200000. Income from Capital Gain of ₹150000 and income from other sources is ₹250000. Contribution to PF is ₹40000, PPF `70000, and LIC is ₹40000. Calculate Income Tax Liability of Mr X.

Q. Mr A had Income from Salary of ₹300000. Income from House property rentals comes out to be ₹150000. Income from Capital gains is ₹300000. Income from other sources is ₹50000.

Deduction made under section 80 C

PF	-	₹ 40,000
PPF	-	₹ 40,000
Fixed deposits	-	₹ 30,000
LIC	-	₹ 40,000

Calculate Income Tax liability of Mr. A

About GST (Goods & Service Tax)

Goods and Services Tax which is abbreviated as GST is the form of Tax which has been Imposed by the Government of India at National Level. The GST levied by the Government of India on the sellers, manufacturers and consumers of goods and services at a National Level.

GST is derived from the concept of Value Added Tax which means that it is applied at each stage and the consumer is supposed to pay the GST amount which is charged by last dealer or the supplier in the supply chain.

Different Tax Heads under GST

GST can be categorized in four different heads such as ______________

1. State Goods and Services Tax (SGST) - State Government collects this Tax.
2. Central Goods and Services Tax (CGST) - Central Government collects this Tax.

3. Union Territory Goods and Services Tax (UTGST) - Union Territory Government collects this tax.

4. Integrated Goods and Services Tax (IGST) - It is collected by the Central Government for Inter State transaction. IGST is also levied on import of goods and services into India and an export of goods act side India.

Remember

CGST + SGST : Revenue is shared equally between centre and state.
IGST - Revenue is collected by Central Govt and is shared with the State as the basic of destination of goods.

- **GST calculation formula**

  For calculating GST, following mentioned formula can be used by the tax payer. Following formula helps to calculate net price of the product after application of GST and removing GST as well.

  The formula for GST calculation -

  1. Add GST

  GST Amount = (original cost x GST %)/100

  Net price = original cost + GST amount

  2. Remove GST

  GST Amount = original cost - (original cost x {100/(100+GST%)})

  Net Price = original cost - GST Amount

  Example - Lets assume that a product is sold for Rs. 2000 and GST applicable to that product is 12%.

  Then the net price of the product becomes Rs. 2,000 + 12% of ₹2,000.

  This comes out as ₹2,000 + ₹240 =₹2,240

11.10 BILLS, TARIFF RATES, FIXED CHARGE, SURCHARGE, SERVICE CHARGE

- BILLS

- A bill is a written statement of money that you owe for goods and services. If you bill someone for goods or services you have provided them with, you give or send them a bill stating how much money they owe you for these goods and services.
- Bill, also called as, Invoice or Tab is a commercial document issued by a seller to a buyer, relating to a sale transaction and indicating the products, quantities and agreed prices for the products or the services the seller had provided the buyer.

Example

<table>
<tr><th colspan="7">INVOICE/BILL</th></tr>
<tr><th colspan="7">Company Name</th></tr>
<tr><th colspan="7">123, Wall Street</th></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Invoice No.</td><td></td><td></td><td></td><td></td><td></td><td>Date Terms</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Description</td><td></td><td></td><td></td><td></td><td></td><td>Amount Owed</td></tr>
<tr><td>Invoice Total</td><td></td><td></td><td></td><td></td><td></td><td>[Currency]</td></tr>
</table>

- **Tariff Rates**

  A tariff is a tax imposed by a government on good and services imported from other countries that serves to increase the price and make imports less desirable or atleast less competitive, versus domestic goods and services.

  Tariffs are generally introduced as a means of restricting trade from particular countries or reducing the importation of specific types of goods and services.

  For Example: In order to discourage the purchase of Italian leather handbags, the U.S government could introduce a tariff of 50% that drives the purchase price of those bags so high that the domestic alternatives are much more afferable. The government's hope is that the added cost will make the imported goods much less desirable.

A rate is derived from the data in the tariff of rates, for example, schedules and rating tables. This rate is set forth by a rating organization.

- **Fixed Charge :**

  A fixed charge is any type of expense that recurs on a regular basis, regardless of the volume of business.

  Fixed charge mainly include loan (principal and interest) and lease payments.

- **Interpretation**

  If your business borrows money from the bank, the bank may say it wants to take a fixed charge over a particular asset of your business, for example, your business premises.

  This means if your business stops repaying the bank, it can seize your business premises and sell them to recover the money it owed.

Important

Holders of fixed charges takes precedence over most other creditors if business is liquidated. For example, if your business is in liquidation, a bank holding a fixed charge over your business's property would be entitled to be repaid first from the sale of property. Only money left over from that sale could go to the rest of the creditors. If you have given fixed charge on one of your business's assets, you can't usually sell that asset without permission from creditor who holds fixed charge.

- **Surcharge**

  A surcharge is an extra fee, charge or tax that is added on to the cost of a good or service, beyond its initial price. Surcharge is 'Tax on Tax'. This charge reflects a locality's need to collect the money for extra services, a hike to defray the cost of increased commodity pricing, such as with a fuel surcharge or extra fee on your wireless bill for access to emergency services.

Many Industries, including travel, telecom and cable will add surcharges to offset the cost of higher prices, such as fuel, or regulatory fees Imposed by the government.

Surcharge in relation to Taxes

Surcharge is an additional charge or tax levied on existing tax. It is levied as percentage on the income tax payable as per normal rates. In case, no tax is due for the financial year, then no surcharge is levied.

The revenue earned via surcharge is solely retained by the Centre and is not shared with the States. Collections from surcharge flow into the Consolidated Fund of India.

Service Charge

- A service charge is a fee collected to pay for the services related to the primary product or service being purchased. The charge is usually added at the time of the transaction.
- Many Industries collect service charges including restaurants, banking and travel and tourism. When collected, these charges may cover service rendered to the customer or they may cover administrative or processing costs.
- Service charges are paid directly to the company.

  For example - A concert venue may charge a service fee in addition to the initial price of a ticket at the time of purchase in order to cover the cost of security or for providing the convenience of electronic purchases.
- Service Charges are also called as Service Fees. They go by no. of different names depending on the industry, including booking fees (hotels), security fee (travel), maintenance fee (banking) and customer service fee.

GST

Question : A man buys a LCD at Rs 40,000 at a discount of 20% on printed price and sills it at printed price. If the sales are intra-state and the rate of GST in 12%

find

a) price a t which LCD is bought by retailer

b) price at which the consumer bought LCD

c) GST paid by retailer

d) GST paid by consumer

Ans : Rate of GST = 120% that will be shared by centre and states equally at 6%

cost of LCD = Rs. 40,000

Discount = $\frac{40000 \times 20}{100}$ = Rs. 8,000

Net price of LCD = Rs 40,000 - Rs 8,000 = Rs 32,000

SGST paid by retailer = $\frac{6}{100} \times$ Rs. 32,000 = Rs. 1,920

CGST paid by retailer to wholesaler =

Total GST paid by retailer to wholesaler = Rs 1,920 + Rs. 1920 = Rs 3,840

a) Purchase cost of LCD = Rs. 32,000 + Rs 3,840 = 35,840

b) Retailer sells the LCD at the fixed price

Consumer buys LCD at printed price Rs 40,000

SGST paid by the consumer to the retailer = $\frac{40000 \times 6}{100} \times$ Rs. 2400

CGST paid by the consumer to retailer = $\frac{6}{100} \times$ Rs. 40,000 = Rs. 2,400

Total GST paid by the consumer = Rs 2400 + Rs 2400 = Rs 4800

c) GST paid by the retailer to govt. Rs 4800 - Rs 3840 = Rs 960

d) GST paid by consumer = Rs 4,800

11.11 Calculation and Interpretation of Electricity Bill, Water Supply and other Supply Bills

Electricity Bill - A bill for money owed for electricity used.

- **Calculation and Interpretation**

The Electricity Bill is based upon the usage of appliances in the house over a monthly period.

All the appliances have a wattage label on them which tells what the device uses.

E.g. Toaster is 1 KW, kettle is 2400 Watt, lightbulb is 100 Watt, TV 85 Watts etc.

Most domestic customers are billed in kWh.

Bills are made up of a fixed component - line charges, network charges, which is based upon the size of the main fuse (s) 10, 30. You pay that regardless of any power used.

Then there is usage charge per KWh, (Unit) of cts/kWh.

- **Electricity bill explained through an Example**

TV is used 4 hours a day at 85 Watts/hour = $4 \times 85 = \frac{340}{1000} = 0.34 \text{ Kw / day}$

Toaster is used 15 minutes/day at 1000 Watts/hour = $\frac{15}{16} \times \frac{1000}{1000} = 0.25 \text{ Kwh}$

Kettle is used for 2 hours/day at 2400 Watts = $\frac{2400 \times 2}{1000} = 4.8 \text{kwh / day}$

15 light bulbs for 5 hours / day at 100 watts = $\frac{15 \times 5 \times 100}{1000} = 7.5 \text{kwh / day}$

Total daily usage = 12.89 kwh / day

Monthly usage 30 days x 12.89 kwh/ day = 386.7 kwh / months

Flat rate tariff at 22 cts / kwh = ₹85.07

Line charge Rs. 1.10 / day x 30 days = ₹33.00

So, total bill is ₹118 .07 for the month

Understanding

Domestic electricity bill is calculated after taking the energy consumption unit at your premises. Nowadays, latest electronic hand held devices are being used and the bill collector need not have to feed the units, as it is laser receiver, and once it is put into the meter chamber it collects all the data and information about the consumers and a copy of print out comes from the device as electricity bill, and is handed over to the consumers on the spot . And, the total data of the day for the consumers will be fed to the computer for keeping the records. The Tariff applicable varies from state to state electricity boards. It will have many categories like domestic, commercial and industrial and place of worship and some are divided into sub groups to provide subsidies, from time to time as and when amended by the state governments.

Generally, domestic bill is calculated as per the slabs provided by the individual electricity boards. From 0-50, 51-100, 101- 200 and 200-300 above 301-500, likewise tariffs are made and fixed to control the consumption of power.

These slabs and tariff rates are taught to the billing meters.

Water Bills

- It is the amount one must pay to use water and sewage services each month. Normally water and sewage is provided by a municipality, but this is not always. Water supply bills are usually based upon one's usage , such as those who use more water are charged more.

- **Calculation and Interpretation Of Water Supply Bill**

  The water bill consists of a service charge, consumption charges and 50% of consumption charges as Sewage maintenance charge

To understand, rates of services and consumption charge-

Service Charge	Consumption per month (litres)	**Consumption Charge**
1. ₹40/- per month per connection for Residential premises having Build up area upto 200 sq. meters	0-,6000 7,000-20,000	0 2/-
2. ₹120/- per month per connection for residential premises having Build up area above 200 sq. metres	21,000-30,000 above 30,000	7/- 10/-

Clarification Regarding Service Charge

The Service charge leviable on the domestic consumers is as follows -

1. For a single water connection in a residential premise on a plot the service charges are leviable on the basis of the build up area of the ground floor of the premises, even if there may be a number of floors in the premises.
2. For multiple connections in any residential area /premises on a plot, service charges shall be leviable on each connection on the basis of build up area of the unit, being fed by the respective water connection.

For Example -

1. If on the ground floor of any premises there are more than one connection, then for each connection the service charge will be leviable on the basis of build up area of the premises served by each connection.
2. If there is only one water connection in a premises having multiple floors, the service charge will be leviable only on the build up area of the ground floor of premises against which connection is sanctioned.
3. For multiple water connections on the ground floor, feeding a residential premises on a plot area which has one or more than one floors, Service charge shall be liable against each water connection, as per build up area of ground floor only.

- **Calculate your water bill yourself**

Example - If Build up area of your house is up to 200 Sqmtrs. and -

- On one connection, if consumer consume water up to 6000 litres per month, consumption charges shall be NIL and monthly bill be ₹40/-.
- On one connection, if consumer consume 10,000 litres water per month, monthly bill will be ₹52/-.
- On one connection, if consumer consume 20,000 litres water per month, monthly bill will be ₹82/-.
- On one connection, if consumer consume 21,000 litres water per month, monthly bill will be ₹93/-.
- On one connection, if consumer consume 31,000 litres water per month, monthly bill will be ₹202/-.

Chapter-XII

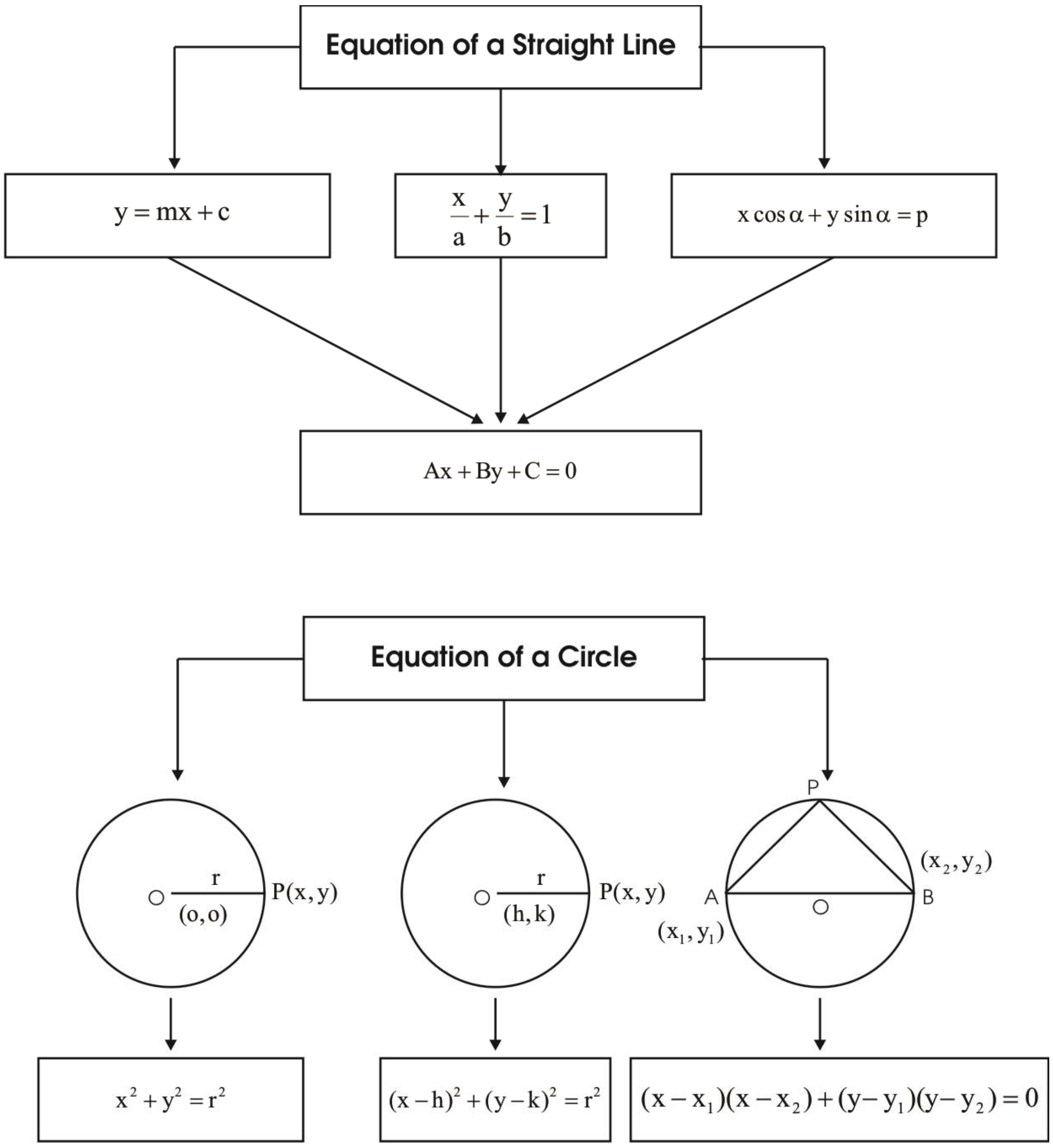
Coordinate Geometry

Learning Objectives:

Students will be able to:

- Form equation of a straight line in different forms.
- Form general equation of a line.
- Reduce the general equation of a line into different forms.
- Find the distance of a point from a line.
- Form equation of a circle in different forms.
- Write the general form of the equation of a circle.
- Form equation of a parabola in standard form.

Concept Map



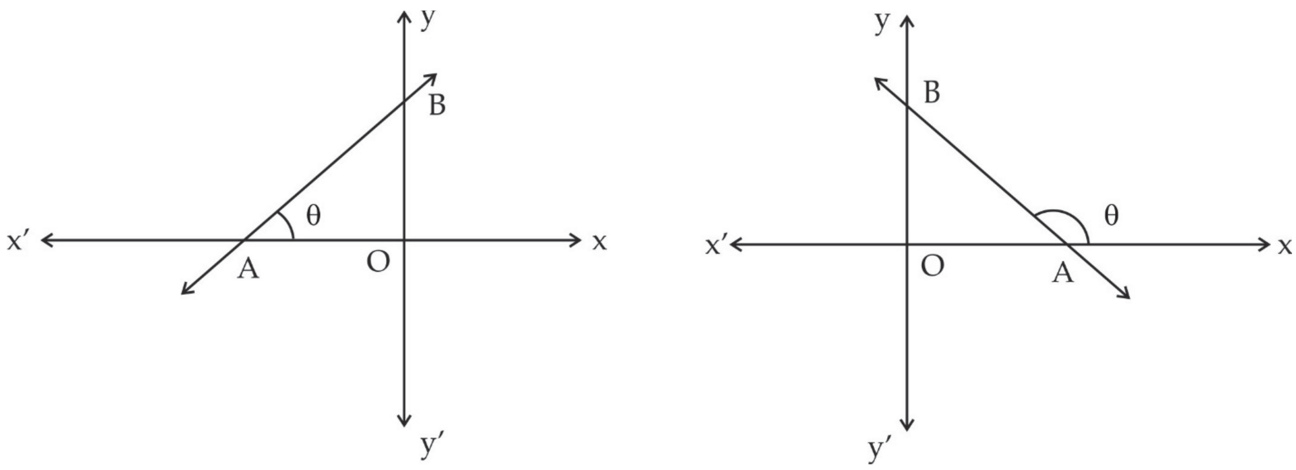
12.1 Straight Line

A straight line is a curve such that every point on the line segment joining any two points on it lies on it.

Equation of a Line: The equation of a straight line is a linear relation between two variables x and y, which is satisfied by the coordinates of each and every point on the line and is not satisfied by any other point in the Cartesian plane.

Slope of a Line:

Inclination of a Line



The inclination of a line is the angle θ which the line makes with the positive direction of the x-axis, measured in anticlockwise direction. Clearly $0^0 \leq \theta \leq 180^0$. If we are keeping θ unique thus $0^0 \leq \theta < 180^0$.

Slope or Gradient of a Line: If θ is the inclination of a non-vertical line, then tan θ is called the slope of the line and is denoted by m.

Remark 1: The slope of a horizontal line is 0, because $\tan 0^0 = 0$.

Remark 2: The slope of a vertical line is not defined, because $\tan 90^0$ is not defined.

Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2 respectively. Let their inclinations be θ_1 and θ_2 respectively.

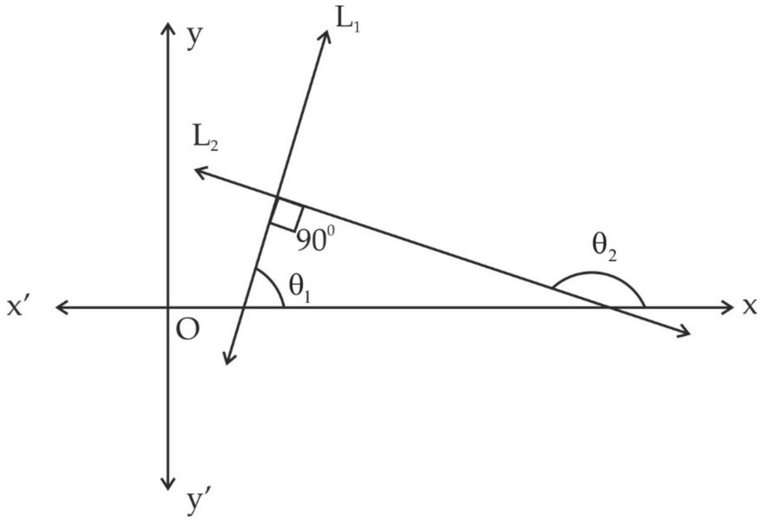
If L_1 and L_2 are parallel

$\Rightarrow \theta_1 = \theta_2$

$\Rightarrow \tan \theta_1 = \tan \theta_2$

$\Rightarrow m_1 = m_2$

12.2 Slopes of Perpendicular Lines



Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2 respectively. Let their inclinations be θ_1 and θ_2 respectively. Then $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

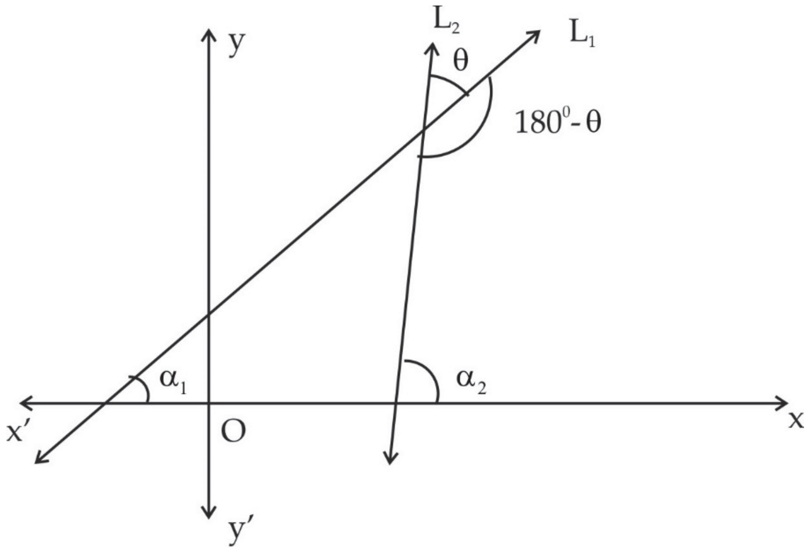
Let L_1 and L_2 be perpendicular to each other. Then $\theta_2 = 90^0 + \theta_1$

$$\Rightarrow \tan\theta_2 = \tan(90^0 + \theta_1) = -\cot\theta_1$$

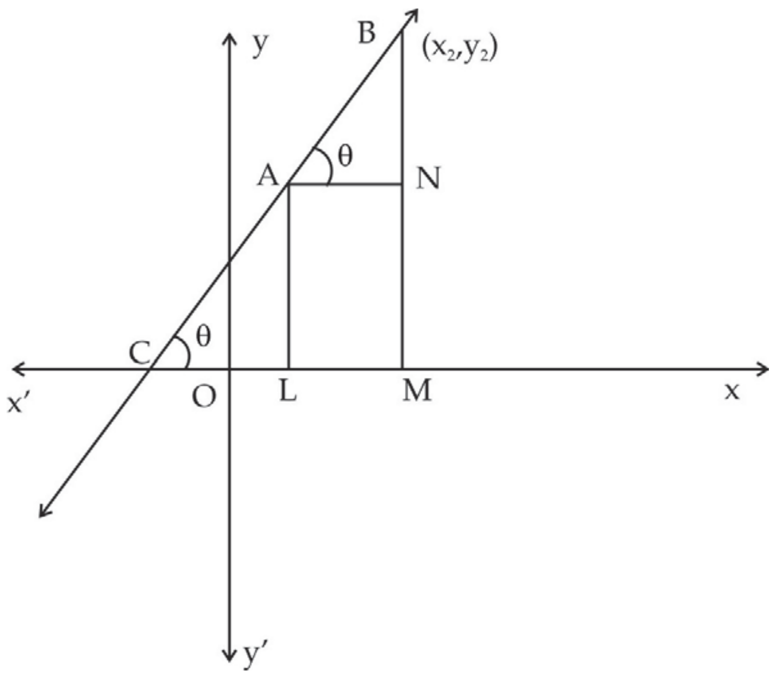
$$\Rightarrow \tan\theta_2 = -\frac{1}{\tan\theta_1}$$

$$\Rightarrow m_2 = -\frac{1}{m_1} \Rightarrow m_1 m_2 = -1$$

Angle between Two Non-vertical and Non-perpendicular Lines



Slope of a Line Passing through Two given Points



Let us consider a non-vertical line CAB, passing through the points A (x_1, y_1) and B (x_2, y_2). Let θ be its inclination Draw AL and BM perpendicular to x-axis and AN $\perp$ BM

Thus $\angle BAN = \angle ACM = \theta$

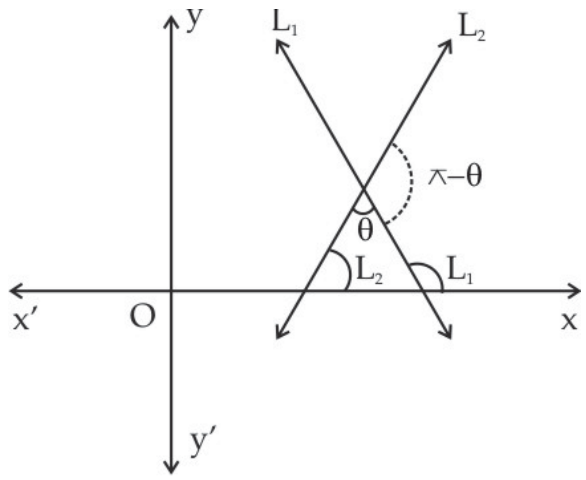
From right triangle ANB, we have

$$\tan\theta = \frac{BN}{AN} = \frac{BM - NM}{LM} = \frac{BM - AL}{OM - OL} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

Hence, the slope of the line joining the points A (x_1, y_1) and B (x_2, y_2) is given by, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

12.3 Angle between two Lines



Let L_1 and L_2 be the two given non-vertical and non-perpendicular lines with slopes m_1 and m_2 respectively. Let α_1 and α_2 be their respective inclinations. Then $m_1 = \tan\alpha_1$ and $m_2 = \tan\alpha_2$.

Let θ and $(180^0 - \theta)$ be the angles between L_1 and L_2.

We have $\theta = \alpha_2 - \alpha_1$ where $\alpha_1, \alpha_2 \neq 90^0$

$$\Rightarrow \tan\theta = \tan(\alpha_2 - \alpha_1) = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \times \tan\alpha_2}$$

$$\Rightarrow \tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} \qquad \text{where } 1 + m_1 m_2 \neq 0$$

Also $\tan(180^0 - \theta) = -\tan\theta = -\dfrac{m_2 - m_1}{1 + m_1 m_2}$

Thus, tangent of the angle between given lines is

$$= \pm\frac{m_2 - m_1}{1 + m_1 m_2}$$

Let $\alpha = \min.\{\theta, (180 - \theta)\}$, then α is acute and so $\tan\alpha > 0$.

$$\therefore \quad \tan\alpha = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

12.4 Various Forms of the Equation of a Line

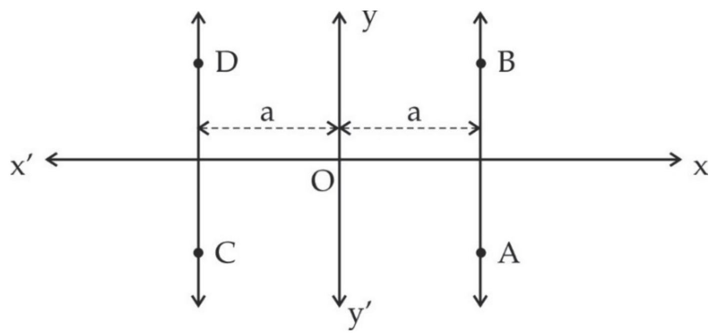
1. **Equation of x-axis:** We know that the ordinate of each point on the x-axis is 0. If P (x, y) is any point on the x-axis, then y = 0.

   Hence the equation of x-axis is y = 0.

2. **Equation of y-axis:** We know that the abscissa of each point on the y-axis is 0. If P (x, y) is any point on the y-axis, then x = 0.

   Hence the equation of y-axis is x = 0.

3. **Equation of a line parallel to y-axis.**

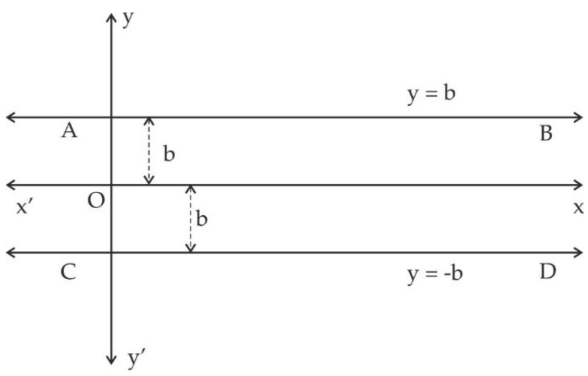


Let AB be a straight line parallel to the y-axis lying on its right hand side at a distance a from it. Thus, the abscissa of each point on AB is a.

If P (x, y) is any point on AB, then x = a. Thus, the equation of a vertical line at a distance a from the y-axis, lying on its right hand side is x = a.

Similarly, the equation of a vertical line at a distance a from the y-axis, lying on its left-hand side is x = –a.

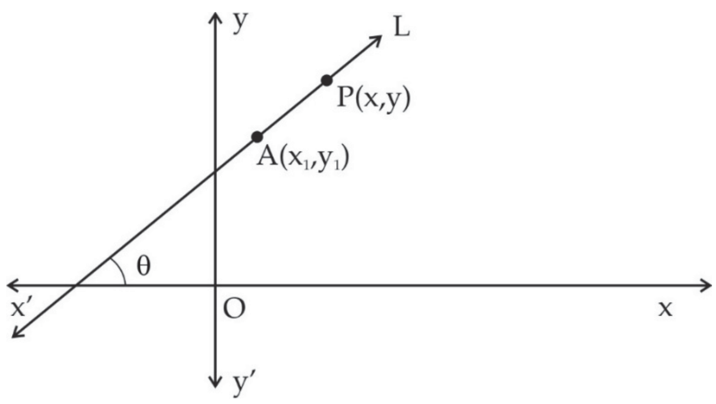
4. **Equation of a line parallel to x-axis**



Let AB be a straight line parallel to the x-axis, lying above it at a distance b from it. Then ordinate of each point on AB is b. If P (x, y) is any point on AB, then y = b. Thus the equation of a horizontal line at a distance b from the x-axis, lying above the x-axis is y = b.

Similarly, the equation of a horizontal line at a distance b from the x-axis, lying below the x-axis is y = –b.

Equation of a Line in Point-Slope Form



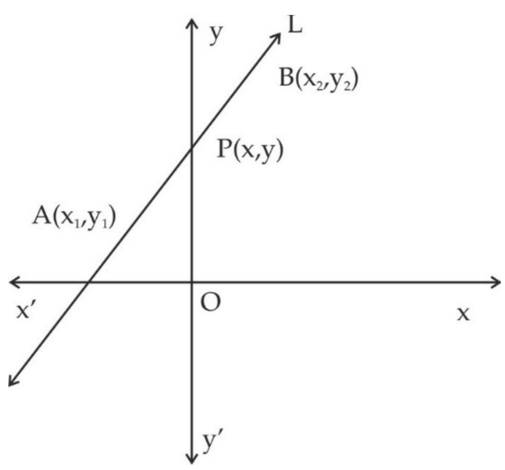
Let L be a non-vertical line with slope m and passing through a point A (x_1, y_1).

Let P (x, y) be an arbitrary point on L.

Then slope of the line L = $\frac{y-y_1}{x-x_1}$

$\therefore$ m = $\frac{y-y_1}{x-x_1}$ $\Rightarrow$ $y-y_1 = m(x-x_1)$ where $m = \tan\theta$

Equation of a Line in Two-Point Form



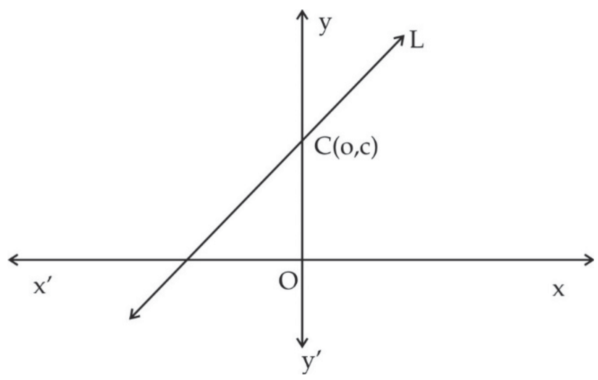
Let L be the line passing through the points A (x_1, y_1) and B (x_2, y_2) and let P (x, y) be an arbitrary point on L. Then slope of AP = slope of AB.

$$\Rightarrow \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

which is the required equation.

Equation of a Line in Slope-Intercept Form



Let L be a given line with slope m, making an intercept c on the y-axis. Then, L cuts the y-axis at the point (O,C).

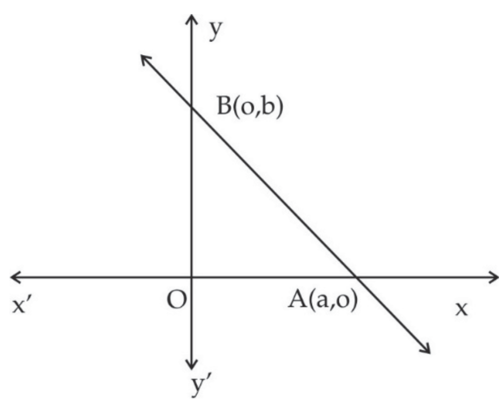
(Remember C could be –ve, O or +ve. The y intercept of the line is C.)

Thus, the line L has slope m and it passes through the point C (o, c).

So by point-slope form, the equation of L is

$$y - c = m(x - 0) \Rightarrow y = mx + c$$

Equation of a Line in Intercept Form



Let AB be the line cutting the x-axis and the y-axis at A (a, o) and B (o, b) respectively. Here, x – intercept and y – intercept of the line are a, b respectively. Let us assume that none of a, b is 0.

So, the equation of line AB is given by

$$\frac{y-0}{x-a}=\frac{b-0}{0-a}$$

$$\Rightarrow \frac{y}{x-a}=\frac{b}{-a}$$

$\Rightarrow$ b (x – a) = – ay

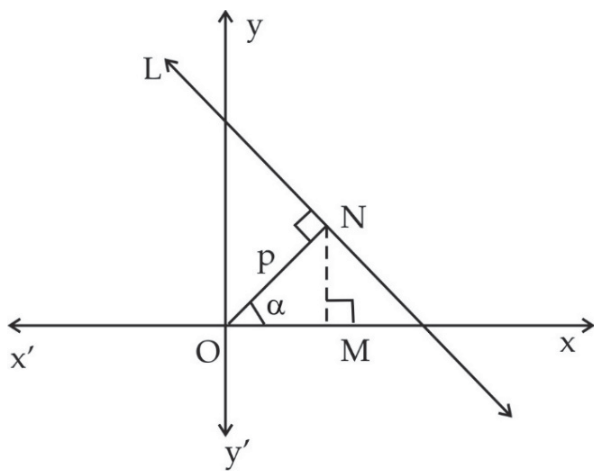
$\Rightarrow$ bx + ay = ab

$$\Rightarrow \frac{x}{a}+\frac{y}{b} = 1$$

(Division of both sides by ab is possible as none of a and b is 0)

Thus, the equation of a line making intercepts a and b on the x and y axis respectively is $\frac{x}{a}+\frac{y}{b}=1$.

Equation of a Line in Normal Form



Let L be the given line. Draw ON perpendicular to L. Let ON = p and let the angle between positive direction of the x-axis and ON be α in anti-clockwise sense.

Draw perpendicular NM on the x-axis. We have

$$\frac{OM}{ON} = \cos\alpha \text{ and } \frac{NM}{ON} = \sin\alpha$$

(In this figure α is acute. But, α could be such that $0 \leq \alpha < 2\pi$) In all possible cases, the co-ordinates of N are ($p \cos\alpha, \sin\alpha$)

$$(\alpha \neq \frac{\pi}{2}, \alpha \neq \frac{3\pi}{2})$$

$\Rightarrow$ $OM = p \cos\alpha$ and $NM = p \sin\alpha$

Thus the coordinates of N are ($p \cos\alpha, p \sin\alpha$)

Moreover, line L is perpendicular to ON.

$\therefore$ Slope of the line L $= \dfrac{-1}{\text{Slope of ON}} = \dfrac{-1}{\tan\alpha} = \dfrac{-\cos\alpha}{\sin\alpha}$

Thus, the line L passes through the point N ($p \cos\alpha, p \sin\alpha$) and has slope

$$m = -\frac{\cos\alpha}{\sin\alpha}$$

So, the equation of the line L is given by

$$y - p\sin\alpha = -\frac{\cos\alpha}{\sin\alpha}(x - p\cos\alpha)$$

$$\Rightarrow (y - p\sin\alpha)\sin\alpha = -\cos\alpha\,(x - p\cos\alpha)$$

$$\Rightarrow x\cos\alpha + y\sin\alpha = p(\cos^2\alpha + \sin^2\alpha)$$

$$\Rightarrow x\cos\alpha + y\sin\alpha = p$$

which is the required equation.

Theorem: Prove that the equation $Ax + By + C = 0$ always represents a line provided A and B are not simultaneously zero.

Proof: Let $Ax + By + C = 0$ where $A^2 + B^2 \neq 0$

Case I: When $A = 0$ and $B \neq 0$

In this case, the given equation becomes

$$By + C = 0 \Rightarrow y = -\frac{C}{B}$$

which represents a line parallel to the x-axis

Case II: When $A \neq 0$ and $B = 0$

In this case, the given equation becomes

$$Ax + C = 0 \Rightarrow x = -\frac{C}{A}$$

which represents a line parallel to the y-axis

You may note that even if $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, the equation would be of the form y = p or – y = p which is of the form $x \cos = \frac{\pi}{2} + y \sin \frac{\pi}{2} = p$ or $x \cos \frac{3\pi}{2} + y \sin \frac{3\pi}{2} = p$

Case III: When $A \neq 0$ and $B \neq 0$

In this case we have

$$Ax + By + C = 0$$

$$\Rightarrow By = -Ax + (-C)$$

$$\Rightarrow y = \left(-\frac{A}{B}\right) x + \left(-\frac{C}{B}\right)$$

This is clearly an equation of a line in slope intercept form y = mx + c,

where $m = \left(-\frac{A}{B}\right)$ and $\left(-\frac{C}{B}\right)$

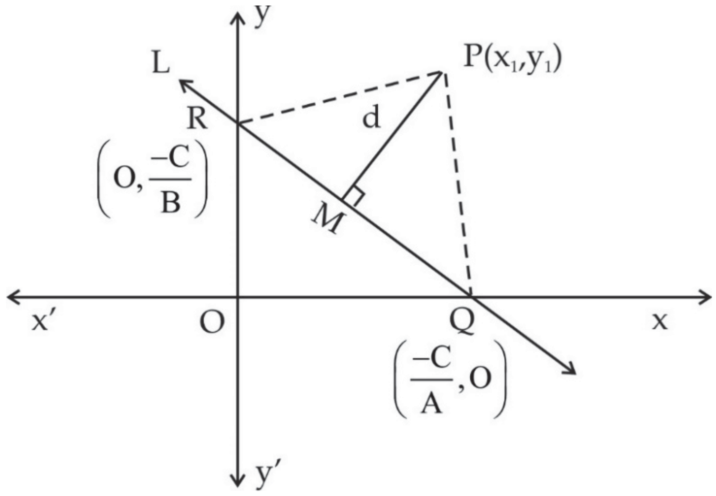
Thus $Ax + By + C = 0$ where $A^2 + B^2 \neq 0$ always represents a line.

Theorem: Prove that the length of perpendicular from a given point P (x_1, y_1) on a line $Ax + By + C = 0$ is given by

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Proof: Let L be the line represented by the equation $Ax + By + C = 0$ and let P (x_1, y_1) be a given point outside it.

Let us assume that none of A, B, C is 0



Let this line L intersect the x-axis and the y-axis at points Q and R respectively, and let PM ⊥ QR.

Now, $Ax + By + C = 0 \Rightarrow Ax + By = -C$

$$\Rightarrow \frac{Ax}{-C} + \frac{By}{-C} = 1$$

$$\Rightarrow \frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1$$

Thus, the given line intersects the x-axis and the y-axis at the points Q $\left(-\frac{C}{A}, 0\right)$ and R $\left(0, -\frac{C}{B}\right)$ respectively.

Area of Δ PQR $= \frac{1}{2}\left|x_1\left(\frac{-C}{B}-0\right)+0.(0-y_1)+\left(\frac{-C}{A}\right).\left(y_1+\frac{C}{B}\right)\right|$

$$= \frac{1}{2}\left|-C\left(\frac{x_1}{B}+\frac{y_1}{A}+\frac{C}{AB}\right)\right|$$

$$= \frac{1}{2}\left|\frac{C}{AB}(Ax_1+By_1+C)\right|$$

$$= \left|\frac{C}{2AB}(Ax_1+By_1+C)\right| \qquad (1)$$

Also, area of Δ PQR = $\frac{1}{2} \times QR \times PM$

$$= \frac{1}{2} \times PM \times \sqrt{(OQ)^2 + (OR)^2}$$

$$= \frac{1}{2} \times PM \times \sqrt{\left(-\frac{C}{A}\right)^2 + \left(-\frac{C}{B}\right)^2}$$

$$= \frac{1}{2} \times PM \times \sqrt{\frac{C^2}{A^2} + \frac{C^2}{B^2}} \qquad (2)$$

From equation (1) and (2), we get

$$\frac{1}{2} \times PM \times \sqrt{\frac{C^2}{A^2} + \frac{C^2}{B^2}} = \left|\frac{C}{2AB}\left(Ax_1 + By_1 + C\right)\right|$$

$$\Rightarrow \frac{1}{2} \times \left|\frac{C}{AB}\right| \times \sqrt{A^2 + B^2} \times PM = \left|\frac{C}{2AB}\right| \left|\left(Ax_1 + By_1 + C\right)\right|$$

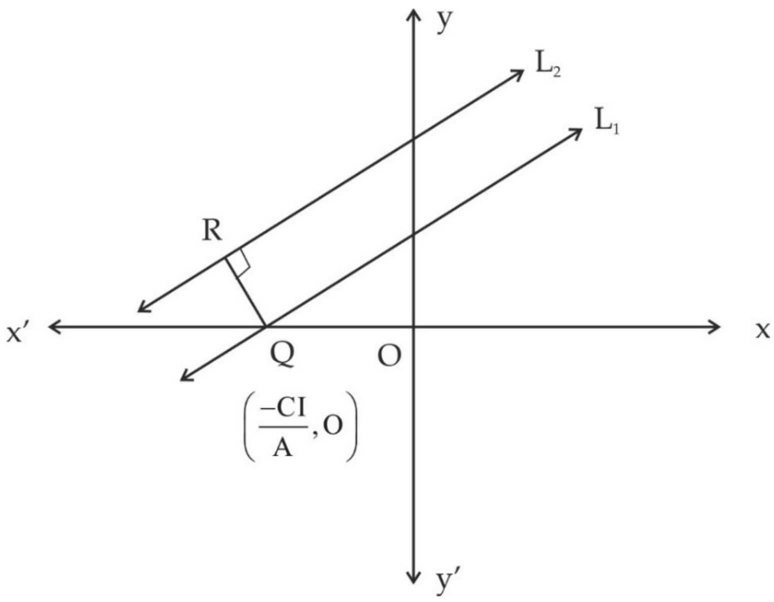
$$\Rightarrow PM = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Hence, $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$

It may be easily seen that the formula stands true even if A = 0, B $\neq$ 0, C = 0 or A $\neq$ 0, B = 0, C = 0 or A = 0, B $\neq$ 0, C $\neq$ 0 or A $\neq$ 0, B = 0, C $\neq$ 0.

Remarks: The length of perpendicular from the origin (0, 0) on the line Ax + By + C = 0 is $\dfrac{|C|}{\sqrt{A^2 + B^2}}$

Distance between Two Parallel Lines



Theorem: Prove that the distance between two parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is given by

$$d = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}$$

Solution: Let L_1 and L_2 be two given parallel lines represented by

$$Ax + By + C_1 = 0 \qquad (1)$$

and $$Ax + By + C_2 = 0 \qquad (2)$$

Putting $y = 0$ in equation (1), we get $x = -\frac{C_1}{A}$ (assuming $A \neq 0$)

Thus L_1, intersects the x-axis at $Q = \left(-\frac{C_1}{A}, 0\right)$

Let $QR \perp L_2$ and let $d = |QR|$. Then

d = length of perpendicular from Q $\left(-\frac{C_1}{A}, 0\right)$ on $Ax + By + C_2 = 0$

$$\Rightarrow \ d = \frac{\left|A \times \left(-\frac{C_1}{A}\right) + B \times 0 + C_2\right|}{\sqrt{A^2 + B^2}} = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}$$

Illustrative Examples

Microeconomic applications: Demand curve

Example 1: Let us assume that the demand curve is described by the line q = mp + b. Find its equations given that a promoter discovers that the demand for theatre tickets is 1200 when the price is Rs. 400, but decreases to 900 when the price is raised to Rs. 450.

Solution: The form of the equation q = mp + b indicates that the price p, is the independent variable and the quantity q, is the dependent variable. The problem gives us two points of the demand line as (400, 1200) and (450, 900). We must identify the slope and the y-intercept of the line.

Slope m = $\frac{\Delta q}{\Delta p} = \frac{q_2 - q_1}{p_2 - p_1} = \frac{900 - 1200}{450 - 400} = \frac{-300}{50}$ = –6

The equation must, therefore, take on the form, q = –6p + b.

Now since (400, 1200) is a point on the demand curve it must satisfy the equation q = –6 p + b. By substitution, we obtain

1200 = – 6 (400) + b

or 1200 = – 2400 + b

b = 3600

Now, since m = – 6 and b = 3600, the equation of the demand line is

q = – 6 p + 3600

Example 2: The equilibrium quantity and the equilibrium price of a product are determined by the point where the supply and demand curves intersect. For a given product, the supply is determined by the line.

q supply = 30 p – 45

and for the same product, the demand is determined by the line

q demand = – 15 p + 855

Determine the equilibrium price and the equilibrium quantity and trace the supply and demand lines on the same graph.

Solution: We must determine the coordinates of point (q, p) situated at the intersection of the two lines. This point must, therefore, satisfy both the supply and the demand equations. The solution to this problem is to solve:

$$q = 30\,p - 45 \qquad (1)$$

$$q = -\,15\,p + 855 \qquad (2)$$

Thus from equation (1) and (2)

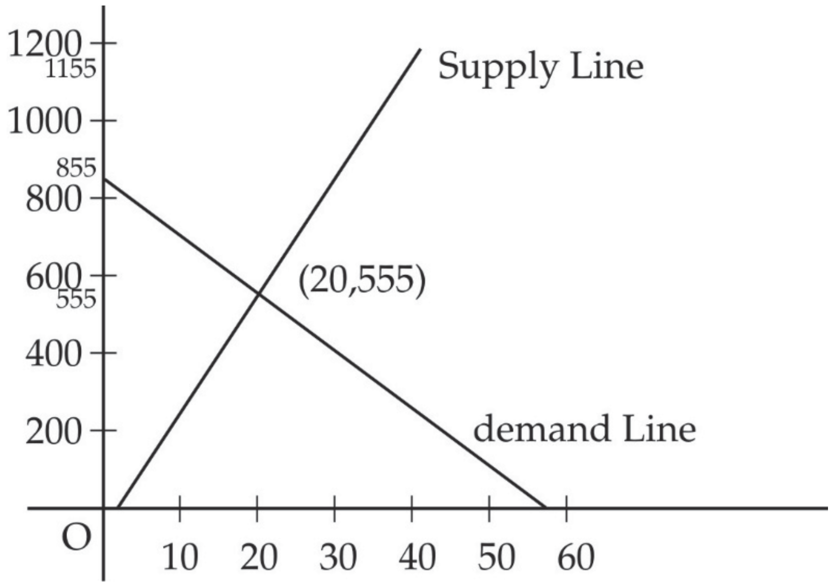
$$30\,p - 45 = -\,15\,p + 855$$

or $\quad 45\,p = 900$

$$p = 20$$

and $\quad q = 30\,(20) - 45 = 555$

The equilibrium price and the equilibrium quantity are, therefore, Rs. 20 and 555 resepctively.



Example 3: Find the equations of the lines parallel to the axes and passing through the point (–3, 5).

Solution: We know that

(i) The equation of a line parallel to the x-axis and passing through (–3, 5) is $y = 5$

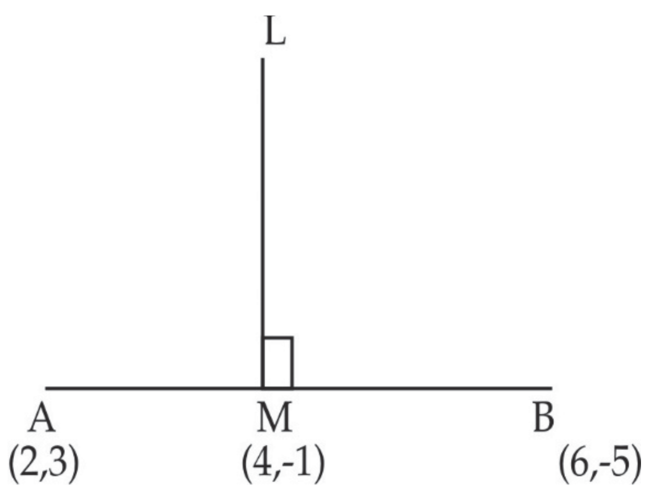
(ii) The equation of a line parallel to the y-axis and passing through (–3, 5) is $x = -3$

Example 4: Find the equation of the perpendicular bisector of the line segment joining the points A (2, 3) and B (6, –5)

Solution: Here A (2, 3) and B (6, –5) are the end points of the given line segment. The midpoint of AB is

$M\left(\frac{2+6}{2}, \frac{3-5}{2}\right)$ i.e., M (4, –1)

Slope of AB = $\frac{-5-3}{6-2} = -2$



Let $LM \perp AB$ and let the slope of LM be m.

Thus, $m \times (-2) = -1 \Rightarrow m = \frac{1}{2}$

Now, LM is the perpendicular bisector of AB with slope = $\frac{1}{2}$ and passing through M (4, –1).

So, the required equation is

$$y + 1 = \frac{1}{2}(x - 4)$$

or $\quad 2y + 2 = x - 4$

$\Rightarrow \quad x - 2y - 6 = 0$

Example 5: Find the equation of the line whose y-intercept is –3 and which is perpendicular to the line $3x - 2y + 5 = 0$.

Solution: $3x - 2y + 5 = 0 \Rightarrow 2y = 3x + 5$

$$\Rightarrow y = \frac{3}{2}x + \frac{5}{2}$$

$\therefore$ Slope of the given line $= \frac{3}{2}$

$\therefore$ Slope of a line perpendicular to the given line

$$= -\frac{2}{3} \qquad \left(\because m_2 = -\frac{1}{m_1}\right)$$

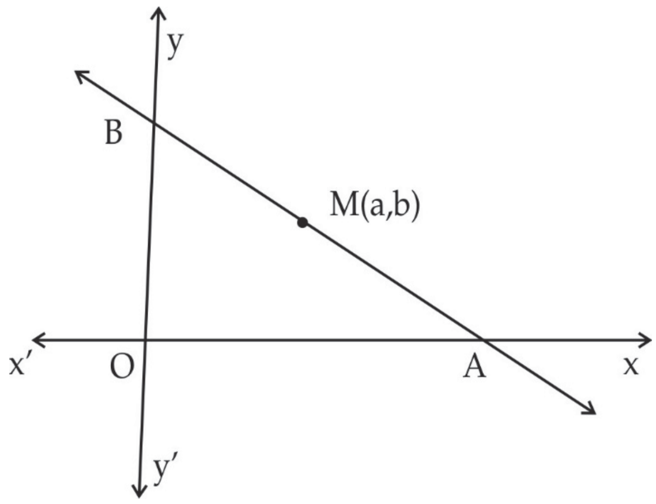
Now, the equation of a line with slope $-\frac{2}{3}$ and y-intercept –3 is given by

$$y = -\frac{2}{3}x - 3$$

$\Rightarrow 2x + 3y + 9 = 0$ which is the required equation.

Example 6: If M (a, b) is the midpoint of a line segment intercepted between the axes, show that the equation of the line is $\frac{x}{a} + \frac{y}{b} = 2$.

Solution: Let the equation of the line be $\frac{x}{c} + \frac{y}{d} = 1$



Thus, it cuts the x-axis and y-axis at the points A (c, 0) and B (o, d). M (a, b) is the midpoint of AB. So $\frac{c+0}{2}$ = a and $\frac{0+d}{2}$ = b.

$\Rightarrow$ c = 2a and d = 2b.

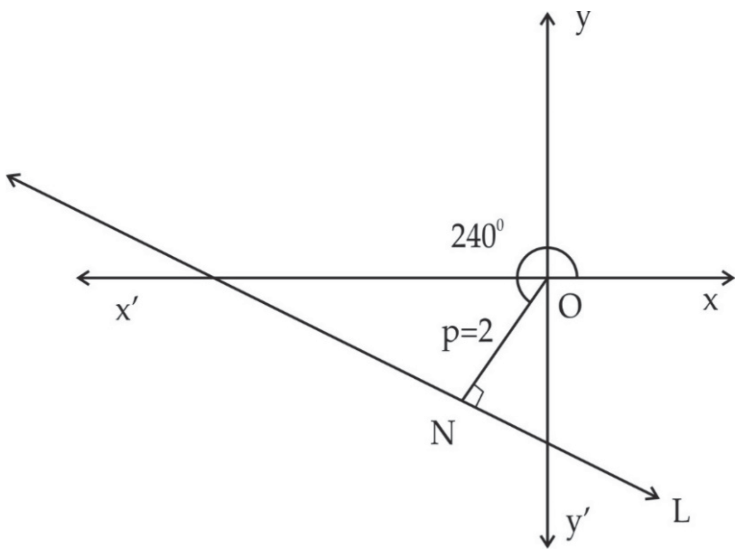
So the required equation is

$$\frac{x}{2a}+\frac{y}{2b}=1 \quad \Rightarrow \frac{x}{a}+\frac{y}{b}=2$$

Example 7: Find the equation of a line whose perpendicular distance from the origin is 2 units and the angle between the perpendicular segment and the positive direction of the x-axis is 240^0.

Solution: Let equation of the line be x cos α + y sin α = p

Here p = 2 units and $\alpha = 240^0$



$\cos 240^0 = \cos(180^0 + 60^0) = -\cos 60^0$

$$= -\frac{1}{2}$$

$\sin 240^0 = \sin(180^0 + 60^0) = -\sin 60^0$

$$= \frac{-\sqrt{3}}{2}$$

So, the equation of the line is

$x \cos 240^0 + y \sin 240^0 = 2$

$$\Rightarrow x \times \left(-\frac{1}{2}\right) + y \times \left(-\frac{\sqrt{3}}{2}\right) = 2$$

$$\Rightarrow x + \sqrt{3}y + 4 = 0$$

which is the required equation.

Example 8: Reduce the equation $\sqrt{3}x + y + 2 = 0$ to the normal form $x \cos \alpha + y \sin \alpha = p$ and hence find the value of α and p.

Solution: We have

$$\sqrt{3}x + y + 2 = 0 \Rightarrow -\sqrt{3}x - y = 2$$

$$\left(-\frac{\sqrt{3}}{2}\right)x + \left(-\frac{1}{2}\right)y = 1$$

Normal form of the equation is $x \cos \alpha + y \sin \alpha = p$

Here, $\cos \alpha = -\frac{\sqrt{3}}{2}$, $\sin \alpha = -\frac{1}{2}$, $p = 1$

Since $\cos \alpha < 0$ and $\sin \alpha < 0$, α lies in the third quadrant.

Now, $\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \left(-\frac{1}{2}\right) \times \left(-\frac{2}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} = \tan(180^0 + 30^0)$

$= \tan 210^0 \Rightarrow \alpha = 210^0$

Thus, $\alpha = 210^0$ and p = 1

Hence, the given equation in normal form is x cos 210^0 + y sin 210^0 = 1.

Example 9: Find the distance between the parallel lines.

$$15x + 8y - 34 = 0 \text{ and } 15x + 8y + 31 = 0$$

Solution: Converting each of the given equation to the form y = mx + c, we get

$$15x + 8y - 34 = 0 \Rightarrow y = -\frac{15}{8}x + \frac{17}{4} \qquad (1)$$

$$15x + 8y + 31 = 0 \Rightarrow y = -\frac{15}{8}x - \frac{31}{8} \qquad (2)$$

Clearly, the slopes of the given lines are equal and so they are parallel.

The given lines are of the form $y = mx + c_1$ and $y = mx + c_2$ where $m = -\frac{15}{8}$, $c_1 = \frac{17}{4}$ and $c_2 = -\frac{31}{8}$

$\therefore$ Distance between the given lines

$$= \frac{|c_2 - c_1|}{\sqrt{1+m^2}}$$

$$= \frac{\left|-\frac{31}{8} - \frac{17}{4}\right|}{\sqrt{1+\left(-\frac{15}{8}\right)^2}} = \frac{\left|-\frac{65}{8}\right|}{\sqrt{1+\frac{225}{64}}} = \frac{\frac{65}{8}}{\sqrt{\frac{289}{64}}}$$

$$= \frac{65}{8} \times \frac{8}{17} = \frac{65}{17} \text{ units}$$

Hence, the distance between the given lines is $\frac{65}{17}$ units.

Exercise 1

1. Find the equation of a line which is equidistant from the lines y = 8 and y = –2.

2. If A (1, 4), B (2, –3) and C (–1, –2) are the vertices of a ΔABC, then find the equation of

   (i) the median through A

   (ii) the altitude through A

   (iii) the perpendicular bisector of BC

3. Find the equation of the bisector of the angle between the coordinate axes.

4. Find the equation of the line passing through the point (2, 2) and cutting off intercepts on the axes, whose sum is 9.

5. Find the equation of the line which is at a distance of 3 units from the origin such that $\tan \alpha = \frac{5}{12}$, where α is the acute angle which this perpendicular makes with the positive direction of the x-axis.

6. Reduce the equation 5x – 12y = 60 to intercept form. Hence find the length of the portion of the line intercepted between the axes.

7. Reduce the equation x + y – 2 = 0 to the normal form.

8. What are the points on the x-axis whose perpendicular distance from the line $\frac{x}{3}+\frac{y}{4}$ = 1 is 4 units.

9. A company produces shoes. When 30 shoes are produced the total cost of production is Rs. 1500. When 50 shoes are produced the costs increases to Rs. 2000. What is the cost equation (C) if it varies linearly in function to the number of shoes produced (q).

10. Consider a market characterised by the following supply and demand curves:

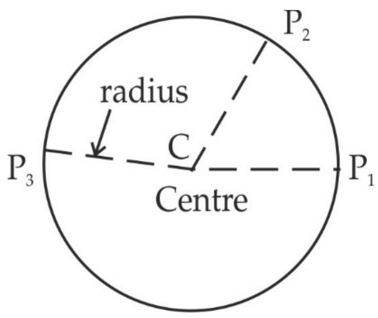
q demand = – 10p + 1000

q supply = 0.2 p + 2986

Find the equilibrium price and quantity.

12.5 Circle

A circle is the set of all points in a plane, each of which is at a constant distance from a fixed point in the plane.



The fixed point is called the centre and the constant distance is called the radius. Radius is always positive.

It $P_1, P_2, P_3, \ldots\ldots$ are points on the circle with center C, and radius r, then

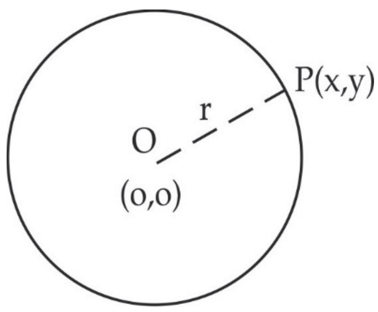
$CP_1 = CP_2 = CP_3 = \ldots\ldots = r$

Equation of a Circle in Standard Form

Let O (0, 0) be the centre of the circle and r be its radius. Let P (x, y) be a point in the plane, then P lies on the circle iff OP = r

i.e., $\sqrt{(x-0)^2+(y-0)^2} = r$

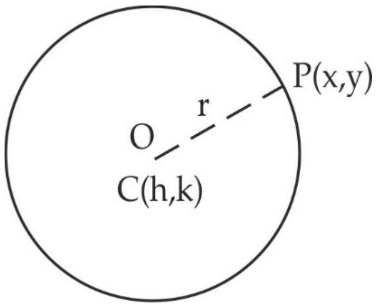
$\Rightarrow x^2 + y^2 = r^2$



This is standard form of the equation of a circle.

Equation of a Circle in Central Form

Let C (h, k) be the centre of the circle and r be its radius. Let P (x, y) be a point in the plane, then P lies on the circle if CP = r

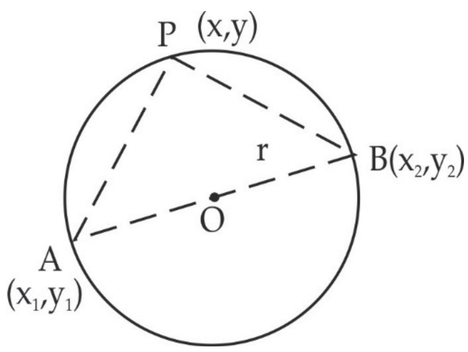


$\Rightarrow \sqrt{(x-h)^2+(y-k)^2} = r$

$\Rightarrow (x-h)^2 + (y-k)^2 = r^2$

This is known as the central form of the equation of a circle.

Equation of a Circle in Diameter Form



Let A (x_1, y_1) and B (x_2, y_2) be the extremities of a diameter of the circle.

Let P (x, y), be any point on the circle. Thus

slope of the line AP = $\frac{y-y_1}{x-x_1}$ and

slope of the line BP = $\frac{y-y_2}{x-x_2}$

Now P lies on the circle, so $\angle APB = 90^0$

The lines i.e. AP and BP are perpendicular to each other.

So $\quad \dfrac{y-y_1}{x-x_1} \cdot \dfrac{y-y_2}{x-x_2} = -1$

$\Rightarrow (y - y_1)\ (y - y_2) = -(x - x_1)(x - x_2)$

$\Rightarrow (x - x_1)\ (x - x_2) + (y - y_1)(y - y_2) = 0$

This is the equation of a circle in diameter form.

Equation of a circle in General Form

We know that the equation of the circle with centre (h, k) and radius r is

$(x - h)^2 + (y - k)^2 = r^2$

$\Rightarrow x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$

It can be written is

$x^2 + y^2 + 2gx + 2fy + c = 0$

where $g = -h$, $f = -k$ and $c = h^2 + k^2 - r^2$

$\therefore \quad r^2 = h^2 + k^2 - c$

$= (-g)^2 + (-f)^2 - c$

$= g^2 + f^2 - c$

But $r^2>0$, so $g^2 + f^2 - c > 0$

Now if we consider any equation

$x^2 + y^2 + 2gx + 2fy + c = 0$

with $g^2 + f^2 - c > 0$ then on adding $g^2 + f^2$ on both the sides, we get

$(x^2 + 2gx + g^2) + (y^2 + 2fg + f^2) + c = g^2 + f^2$

$\Rightarrow \quad (x + g)^2 + (y + f)^2 = g^2 + f^2 - c$

$\Rightarrow \quad \{x - (-g)\}^2 + \{y - (-f)\}^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$

$\Rightarrow \quad \sqrt{\{x-(-g)\}^2 + \{y-(-f)\}^2} = \sqrt{g^2 + f^2 - c}$

The distance of the point (x, y) from the point (–g, –f) is a fixed positive real number r $\left(=\sqrt{g^2+f^2-c}\right)$

This is the set of all points (x, y) which is at a constant distance r $\left(=\sqrt{g^2+f^2-c}\right)$ from the fixed point (–g, –f)

So the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle with centre (–g, –f) and radius $\sqrt{g^2+f^2-c}$. This is general form of the equation of a circle.

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

12.6 Properties of Circles

1. If $g^2 + f^2 - c = 0$, the equation of circle is satisfied by one and only one point (–g, –f). Therefore it represents a single point known as point circle.

2. If $g^2 + f^2 - c < 0$, the equation of the circle is not satisfied by any real value of x, y i.e. it is not satisfied by the co-ordinates of any point in the plane.

3. The general equation of a circle has the following character stics.

   (i) It is an equation of second degree in x, y containing no product term xy.

   (ii) Coefficients of x^2 = Coefficients of y^2 = 1

4. The equation $ax^2 + ay^2 + 2gx + 2fy + c = 0$ represents a circle if $a \neq 0$ and $g^2 + f^2 - c > 0$. Its centre is $\left(-\frac{g}{a}, -\frac{f}{a}\right)$ and radius is $\frac{\sqrt{g^2+f^2-ac}}{|a|}$.

   The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle if

   (i) $a = b \neq 0$

   (ii) $h = 0$ and

   (iii) $g^2 + f^2 - ac > 0$

Illustrative Examples

Example 1: Find the equation of a circle with centre (3, –2) and radius 5.

Solution: We know that the equation of a circle with centre C (h, k) and radius r is given by

$$(x - h)^2 + (y - k)^2 = r^2$$

Here h = 3, k = –2 and r = 5

$\therefore$ Required equation of the circle is

$$(x - 3)^2 + (y + 2)^2 = 5^2$$

$\Rightarrow x^2 + y^2 - 6x + 4y - 12 = 0$

Example 2: Find the equation of the circle with centre (2, 2) and which passes through the point (4, 5).

Solution: The centre of the circle is C (2, 2) and it passes through the point P (4, 5)

$\therefore$ radius of circle = CP = $\sqrt{(4-2)^2 + (5-2)^2}$

$$= \sqrt{4+9}$$

$$= \sqrt{13}$$

$\therefore$ Equation of the circle is

$$(x - 2)^2 + (y - 2)^2 = \left(\sqrt{13}\right)^2$$

$\Rightarrow x^2 + y^2 - 4x - 4y - 5 = 0$

Example 3: Find the equation of the circle of radius 5 whose centre lies on x–axis and passes through the point (2, 3).

Solution: Let equation of the circle be $(x - h)^2 + (y - k)^2 = r^2$ centre lies on x-axis so centre is (h,o) Since the circle passes through A (2, 3) and has radius 5.

$\therefore \quad (2 - h)^2 + (3 - 0)^2 = 5^2$

$\Rightarrow\ (2-h)^2 = 16 \Rightarrow 2-h = 4, -4$

$\Rightarrow h = -2, 6$

$\therefore$ The centre of the circle is (–2, 0) or (6, 0).

Equation of the circle is

$$(x+2)^2 + (y-0)^2 = 5^2 \quad \text{or} \quad (x-6)^2 + (y-0)^2 = 5^2$$

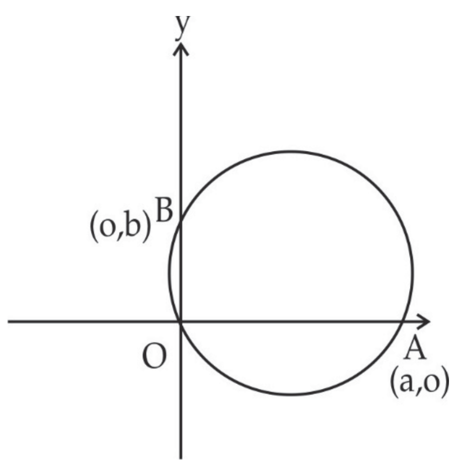
i.e. $\quad x^2 + y^2 + 4x - 21 = 0 \quad$ or $\quad x^2 + y^2 - 12x + 11 = 0$

There are two circles satisfying the given conditions.

Example 4: Find the equation of the circle passing through (0, 0) and which makes intercepts a and b on the coordinate axes.

Solution: Since the circle cuts off intercepts a and b on the coordinate axes, so it passes through the points A (a, 0) and B (o, b).

As the axes are perpendicular to each other and hence AB becomes a diameter of this circle.



$\therefore$ The equation of the circle is $(x-a)(x-0) + (y-0)(y-b) = 0$

or, $\quad x^2 + y^2 - ax - by = 0$

Example 5: Find the centre and radius of the circle $x^2 + y^2 - 8x + 10y - 12 = 0$.

Solution: The given equation is

$$x^2 + y^2 - 8x + 10y - 12 = 0$$

Comparing it with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$g = -4$, $f = 5$ and $c = -12$

$\therefore\ r^2 = g^2 + f^2 - c = (-4)^2 + 5^2 - (-12) = 53 > 0$

Or, $r = \sqrt{53}$

Hence, the given equation represents a circle with centre $(-g, -f)$ i.e., $(4, -5)$

and radius = $\sqrt{53}$

Exercise 2

Very short answer type questions

1. Find the equation of the circle with:

   (i) centre (0, 2) and radius 2.

   (ii) centre (0, 0) and radius 3.

   (iii) centre $(-a, -b)$ and radius $\sqrt{a^2 - b^2}$.

2. Find the equation of the circle drawn on a diagonal of the rectangle as its diameter whose sides are the lines $x = 4$, $x = -5$, $y = 5$ and $y = -1$.

3. Which of the following equations represent a circle? If so, determine its centre and radius.

   (i) $3x^2 + 3y^2 + 6x - 4y = 1$

   (ii) $2x^2 + 2y^2 + 3y + 10 = 0$

   (iii) $x^2 + y^2 - 12x + 6y + 45 = 0$

4. One end of a diameter of the circle $x^2 + y^2 - 6x + 5y - 7 = 0$ is $(-1, 3)$. Find the coordinates of the other end of the diameter

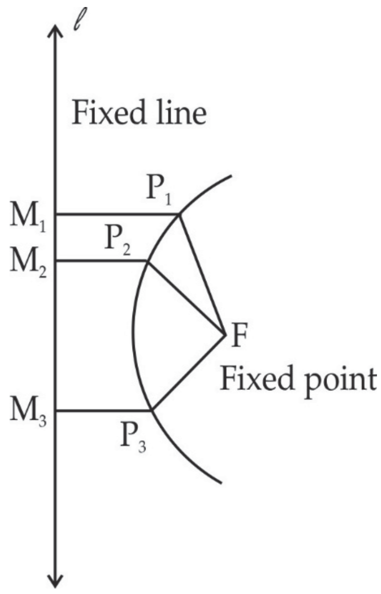
5. Find the equation of the circle passing through the points (2, 3) and (–1, 1) and whose centre lies on the line $x - 3y - 11 = 0$.

6. Find the value of p so that $x^2 + y^2 + 8x + 10y + p = 0$ is the equation of a circle of radius 7 units.

7. Find the value of k for which the circles $x^2 + y^2 - 3x + ky - 5 = 0$ and $4x^2 + 4y^2 - 12x - y - 9 = 0$ are concentric.

12.7 Parabola

A parabola is the set of all points in a plane which are equidistant from a fixed line and a fixed point (not on the line) in the plane.



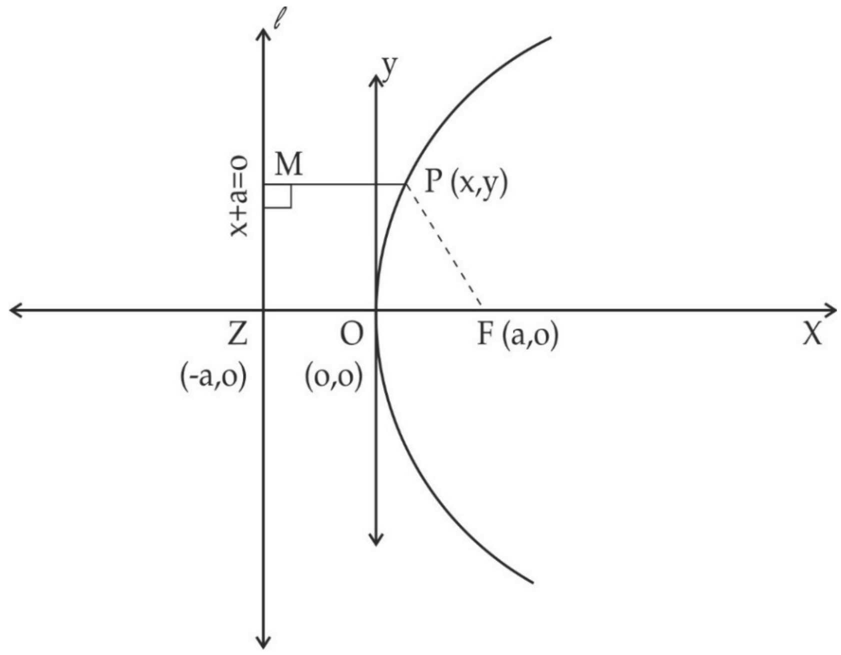
The fixed line (say ℓ) is called the directrix of the parabola and the fixed point (say F) is called the focus of the parabola.

If P_1, P_2, P_3.... are points on the parabola and M_1P_1, M_2P_2, M_3P_3 are perpendiculars to the directrix ℓ, then

$$FP_1 = M_1P_1,\ FP_2 = M_2P_2,\ FP_3 = M_3P_3 \text{ etc.}$$

The line passing through the focus and perpendicular to the directrix is called the axis of the parabola. The point of intersection of parabola with the axis is called the vertex of the parabola. The equation of a parabola is simplest if its vertex is at the origin and its axis lies along either x-axis or y-axis.

Equation of a parabola in the standard form



Let F be the focus, ℓ be the directrix and Z be the foot of perpendicular from F to the line ℓ.

Take ZF as x-axis Let O be the mid-point of ZF. Take O as the origin. Then the line through 0 and perpendicular to ZF becomes y-axis.

Let ZF = 2a (a > 0) then

ZO = OF = a

Since F lies to the right of O and Z to the left of O, coordinates of F and Z are (a, 0) and (–a, 0) respectively. Therefore, the equation of the line ℓ i.e., directrix is x = –a or x + a = 0.

Let P (x,y) be any point in the plane of the line ℓ and the point F, and MP be the perpendicular distance from P to the line ℓ then P lies on the parabola iff

FP = MP

or $$\sqrt{(x-a)^2+y^2}=\frac{|x+a|}{1}$$

or $\quad (x - a)^2 + y^2 = (x + a)^2$

$\Rightarrow \quad x^2 + a^2 - 2ax + y^2 = x^2 + a^2 + 2ax$

$\Rightarrow \quad y^2 = 4ax$

Hence, the equation of a parabola is $y^2 = 4ax$, $a > 0$, with the focus F $(a, 0)$ and directrix $x + a = 0$

It is called standard form of the equation of a parabola.

Some observations about the parabola $y^2 = 4ax$, $a > 0$

If (x, y) is a point on the parabola $y^2 = 4ax$, then $(x, -y)$ is also a point on the parabola

So, the given parabola is symmetrical about x-axis.

(i) This line is the axis of symmetry of the parabola $y^2 = 4ax$.

(ii) The focus is F $(a, 0)$ and directrix is $x + a = 0$.

(iii) The point O $(0, 0)$ where the axis of parabola meets the parabola is the vertex of the parabola.

(iv) If $x < 0$, then $y^2 = 4ax$ has no real solutions in y and so there is no point on the curve with negative x coordinate i.e., on the left of y-axis.

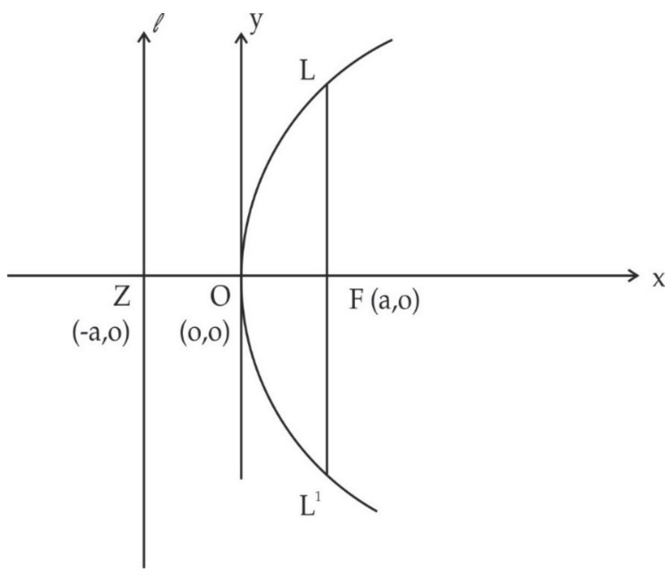
When $x = 0$ we get $y^2 = 0 \Rightarrow y = 0$. Thus $(0, 0)$ is the only point of the y-axis which lies on it. Therefore the entire curve, except the origin, lies to the right of y-axis.

(v) A chord passing through the focus F and perpendicular to the axis of the parabola is called the latus rectum of the parabola.

12.8 Length of the Latus Rectum

Let chord LL′ be the latus rectum of the parabola, then LL′ passes through focus F $(a, 0)$ and is perpendicular to x-axis.

Let LF = k $(k>0)$, then the points L and L′ are (a, k) and $(a, -k)$ respectively.



As L (a, k) lies on the parabola $y^2 = 4ax$, we get

$$k^2 = 4a \times a \Rightarrow k = 2a$$

$\therefore$ The points L and L′ are (a, 2a), (a, –2a) and length of the latus rectum = LL′= 2k = 4a. Also equation of the latus rectum is x = a or x – a = 0.

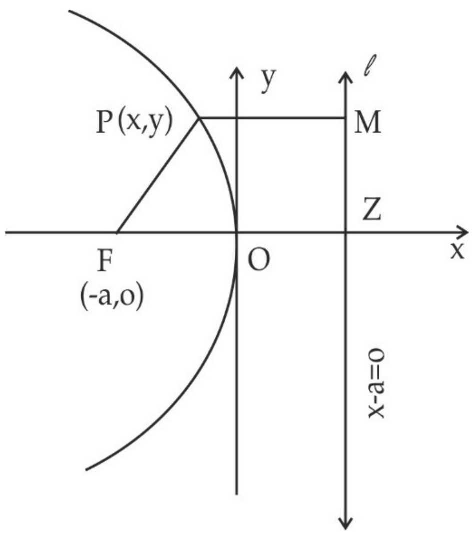
To find the other equations of a parabola in standard form

(i) Focus F (–a, 0), a > 0 and the line x – a = 0 as directrix.

(ii) Focus F (0, a), a > 0 and the line y + a = 0 as directrix.

(iii) Focus (0, –a), a > 0 and the line y – a = 0 as directrix.

(i) Let P (x, y) be any point in the plane of directrix and focus and MP be the perpendicular distance from P to the directrix.



Then P lies on parabola.

Iff FP = MP

$\Rightarrow \sqrt{(x+a)^2 + y^2} = |x-a|$

$\Rightarrow (x + a)^2 + y^2 = (x - a)^2$

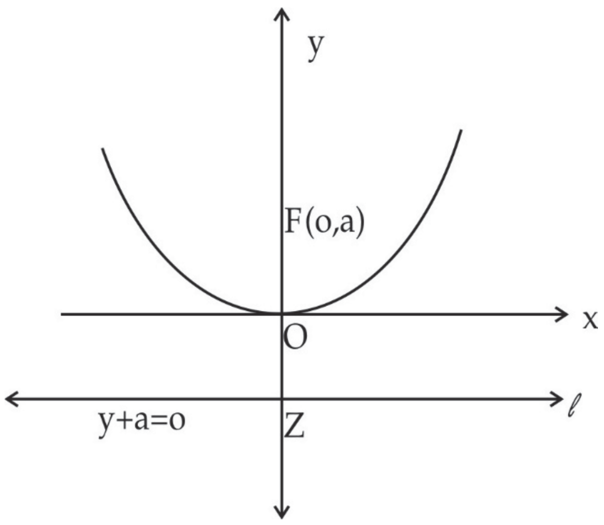
$\Rightarrow x^2 + a^2 + 2ax + y^2 = x^2 + a^2 - 2ax$

$\Rightarrow y^2 = -4ax, a > 0$

The axis is y = 0, equation of directrix x – a = 0, focus (–a, 0). Length of latus rectum = 4a, equation of latus rectum x + a = 0.

(ii) The equation of the parabola opening upwards is $x^2 = 4ay, a > 0$.

Axis x = 0, directrix y + a = 0, Focus (0, a).



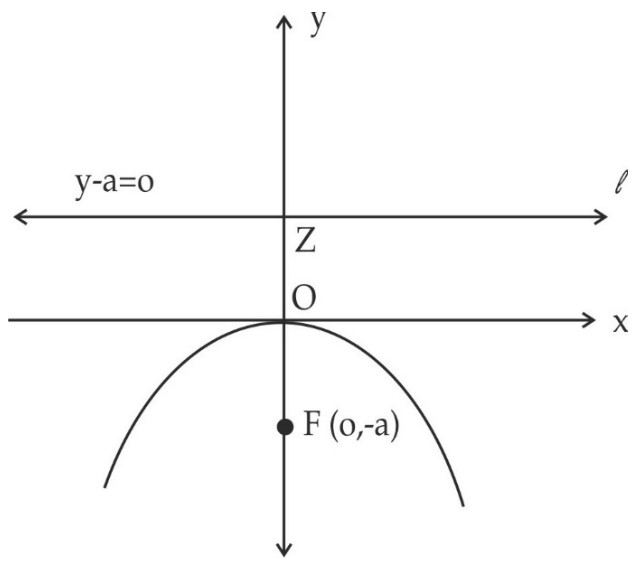
Length of the latus rectum = 4a

Equation of latus rectum is y – a = 0

(iii) The equation of the parabola opening downwards is

$x^2 = -4\,ay, a > 0$

Axis x = 0, directrix y – a = 0, Focus (0,–a).



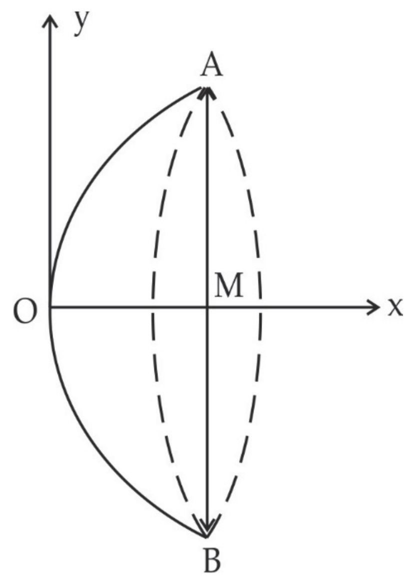
Length of the latus rectum = 4a

Equation of the latus rectum is y + a = 0

Illustrative Examples

Example 1: If a parabolic reflector is 20 cm in diameter and 5 cm deep, find the focus.

Solution: Let AOB be the parabolic reflector which is 20 cm in diameter and 5 cm deep, then AB = 20 cm and OM = 5cm, where M is midpoint of AB.



The equation of the parabola can be taken as $y^2 = 4ax$

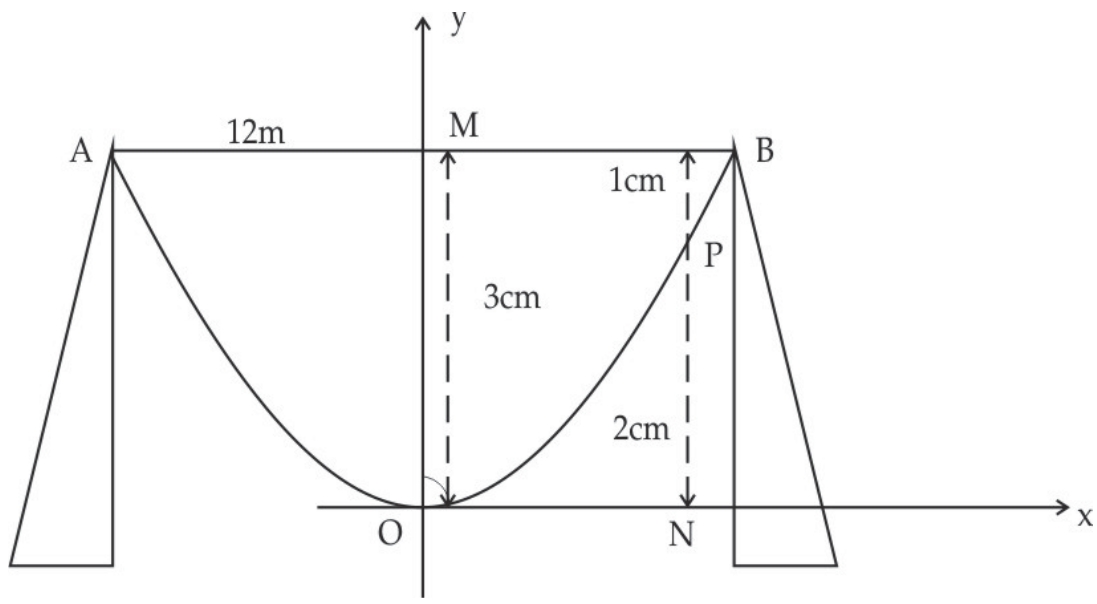
Since the point A (5, 10) lies on the parabola, so

$$10^2 = 4a \times 5 \Rightarrow a = 5$$

$\therefore$ The coordinates of the focus are (a, 0) i.e., (5, 0).

Example 2: A beam is supported at its ends by supports which are 12 metres apart. Since the load is concentrated at the centre of the beam there is a deflection of 3 cm at the centre and the deflected beam is in the shape of a parabola. How far from the centre is the deflection 1 cm?

Solution:



Let AOB be the beam in the shape of a parabola with vertex at the lowest point O. The beam is supported at the points A and B.

Take the vertex O of the parabola as origin and the axis of the parabola as y-axis. The equation of the parabola is $x^2 = 4ay$.

Given AB = 12 m and OM = 3 cm = $\frac{3}{100}$ m.

As M is the midpoint of AB, MB = 6 m.

The coordinates of B are $\left(6, \frac{3}{100}\right)$

Since B lies on the parabola

$$6^2 = 4a \times \frac{3}{100}$$

$$\Rightarrow \; a = \frac{36 \times 100}{12} = 300$$

Let P be any point on the parabola whose deflection is 1 cm, then NP = 2 cm = $\frac{2}{100}$ m. Let ON = x metre, then the coordinates of P are $\left(x, \frac{2}{100}\right)$.

Since P lies on the parabola, we get

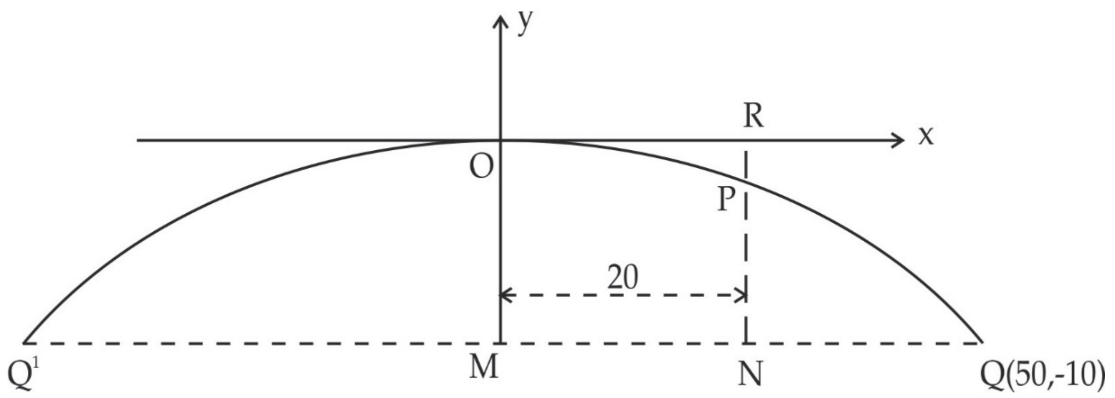
$$x^2 = 4 \times 300 \times \frac{2}{100}$$

$$\Rightarrow \; x^2 = 24 \;\Rightarrow\; x = \pm 2\sqrt{6}.$$

Hence, the points of the beam where the deflection is 1 cm are at a distance of $2\sqrt{6}$ m from the centre.

Example 3: The girder of a railway bridge is a parabola with its vertex at the highest point 10 metre above the ends. If the span is 100 metres, find height of the girder at 20 metres from its mid-point.

Solution:



Take the vertex O of the parabola as origin and the axis of the parabola as y-axis. Therefore, the equation of the parabola is $x^2 = -4ay$.

Given OM = 10 metre and Q'Q = 100 metres

As M is the midpoint of Q'Q, MQ = 50 metres. Therefore coordinates of Q are (50, –10).

Since Q lies on the parabola, so

$$50^2 = -4a(-10) \Rightarrow a = \frac{125}{2}$$

Let MN = 20 metres, draw NR ⊥ MQ to meet the parabola at P.

As P lies below the x-axis, coordinates of P are (20, –p) where p = PR > 0

Since P lies on the parabola, we get

$$20^2 = -4.\frac{125}{2}(-p) \Rightarrow p = \frac{400}{250} = \frac{8}{5} = 1.6$$

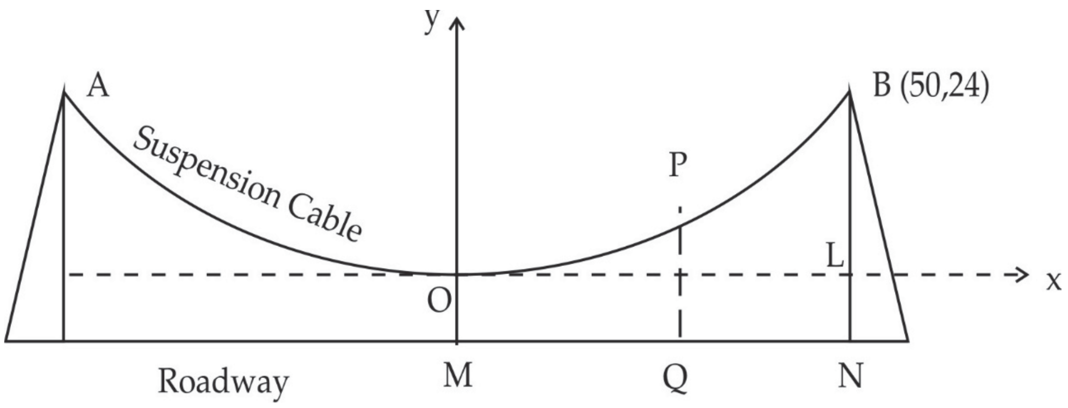
∴ The required height = NP = NR – PR

= 10 – 1.6 as NR = OM = 10 m

= 8.4 metres

Example 4: The cable of a uniformly loaded bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of the supporting wire attached to the roadway 18 m from the middle.

Solution:



The bridge is hung by supporting wires in a parabolic arc with vertex at the lowest point and the axis vertical. The equation of the parabola can be taken as

$x^2 = 4ay$.

Shortest supporting wire = OM = 6 m.

Longest supporting vertical wire = NB = 30 m.

$\therefore$ LB = 30 – 6 = 24 m.

Also OL = $\frac{1}{2}$ × 100 m = 50 m.

$\therefore$ The point B is (50, 24) which lies on the parabola, and so we get

$$(50)^2 = 4a \times 24 \Rightarrow a = \frac{2500}{96}$$

Let PQ be the vertical supporting wire at a distance of 18 m from M. If PQ = k metres, then the point P (18, k – 6) lies on parabola, we get

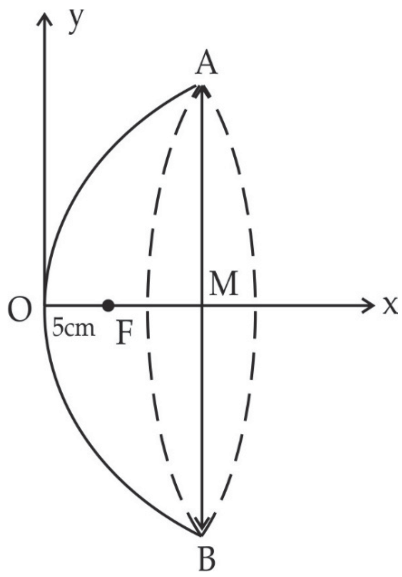
$$18^2 = 4 \times \frac{2500}{96}(k-6) \Rightarrow \frac{324 \times 24}{2500} = k - 6$$

$$\Rightarrow k = \frac{324 \times 24}{2500} + 6 = 9.11 \text{ metres (approx.)}$$

Hence the length of the wire required = 9.11 metres (approx.)

Exercise 3

1. The focus of the parabolic mirror is at a distance of 5 cm from its vertex. If the mirror is 45 cm deep, find the distance AB.



2. An arc is in the form of a parabola with its axis vertical. The arc is 10 m high and 5 m wide at the base. How wide is it 2 m from the vertex of the parabola?

3. The towers of a suspension bridge, hang in the form of a parabola, have their tops 30 metres above the roadway and are 200 metres apart. If the cable is 5 metres above the roadway at the centre of the bridge, find the length of the vertical supporting cable 30 metres from the centre.

4. The girder of a railway bridge is in the form of a parabola with its vertex at the highest point, 15 metres above the ends. If the span is 150 metres, find its height at 30 metres from the midpoint.

5. A water jet from a fountain reaches its maximum height of 4 metres at a distance of 0.5 metres from the vertical passing through the point O of the water outlet. Find the height of the jet above the horizontal OX at a distance 0.75 metre from the point O.

Answer Unit – VIII

Exercise – 1

1. Y = 3
2. (i) $13x-y-9=0$ (ii) $3x-y11=0$ (iii) $3x-y+4=0$
3. $y=\pm x$
4. $x+2y=6$
5. $12x=5y=39$
6. $\frac{x}{12}+\frac{y}{-5}=6$, a = 12, b = -5
7. $x\,Cos\,45^0+y\,Sin45^0=\sqrt{2}$
8. (8, 0), (-2, 0)
9. $c=25+75^0$

10. 68, 76 and 312, 35

Exercise – 2

1. (i) $x^2 + y^2 - 4y = 0$

   (ii) $x^2 + y^2 - 9 = 0$

   (iii) $x^2 + y^2 + 2ax + 2by + 2b^2 = 0 = 0$

2. $x^2 + y^2 + x - 4y - 25 = 0$

3. (i) Circle, Centre $(-1, \frac{2}{3})$, radius = $\frac{4}{3}$

   (ii) Not circle

   (iii) A point circle with centre (6, -3) and radius zero

4. (7, -8)
5. $x^2 + y^2 + 7x + 5y - 14 = 0$
6. -8
7. $-\frac{1}{4}$

Exercise – 3

1. 1.56 m approx.
2. 60 m
3. 2.23 m approx.
4. 7.25 m
5. 12.6 m
6. 13 m

Chapter-XIII

Practical and Project work

Learning Objectives

After completion of this unit the students will be able to

- Know the origin and history of interest rate
- Learn various forms of interest rate
- Comprehend practical applications of interest rate in real life situation.
- Have an outlook of various economic theory associated with interest rate

Practical and Project work

To focus the applied nature of the subject, due emphasis be given for practical and project work.This shall be assessed internally. However this should not be considered in isolation of teaching units. The practical and project work has to be integrated with classroom transactions. Some additional practical works are also incorporated within the units to increase the effectiveness of the teaching learning process.

Weightage of Internal Assessment is of 20 marks. Internal Assessment can be a combination of activities spread throughout semester/ academic year. These include, projects and spreadsheet based practicals. Teachers can choose activities from the suggested list of practical or they can plan activities of similar nature related to the topics detailed in the syllabus. For data based practical, teachers are encouraged to use data from local sources to make it more relevant for students.

Weightage for each area of internal assessment may be as under:

Sr.No.	Area and weightage	Assessment Area	Marks allocated
1	Project work (10 marks)	Project work and record	5
		Year End Presentation/Viva of the Project	5
2	Practical work (10 marks)	Performance of practical and record	5
		Yearend test of any one practical	5
Total			20

Practical: Use of spread sheet

Learning outcomes:

- Students shall be able to :
- calculate average, interest (simple and compound)
- create pictographs
- draw pie chart and bar graphs

- calculate central tendency
- visualizing graphs

Suggested practical using spread sheet

1. Plot the graph of functions on spreadsheet; study the nature of function at various points, drawing lines of tangents;
2. Create budget of income and spending;
3. Create compare sheet of price, features to buy a product;
4. Prepare best option plan to buy a product by comparing cost, shipping charges, tax and other hidden cost;
5. Smart purchasing during sale season;
6. Prepare a report card using scores of last four exams and compare the performance;
7. Collect the data on weather, price, inflation, and pollution. Sketch different types of graphs.

About Project work:

Some suggested project works are given in the syllabus. You may refer the syllabus for details. While selecting the project, please ensure that students apply what they have learnt, in a practical situation. A student may select a topic related to the application of Mathematics in a professional field. The projects may involve, data analysis and conclusions, mathematical modelling, computer programme, Literature review, development of teaching learning tool for a particular concept, identifying applications of a topic, survey and analysis of the results, demonstration of mathematical tools like GeoGebra or simulations, mathematisation of life situations, contextualisation of mathematical results etc. It should not be simply collecting information from net or other sources and preparing a project report. Independent mathematical work should be reflected in the project work.

The project work may be done individually or in a small group. While forming the group it may be ensured that each member of the group contributes in the work.

Therefore a large project can be done by 2-3 students and small project may be completed by individual students. The roles and responsibilities in a group may be assigned clearly to ensure participation of all the members. The project work may be planned and executed in such a way that students will be able to communicate, collaborate and connect with the peer group. Opportunities may be given to individual learners to present their work before the class and teachers may promote open mildness through deliberations among the students. The group activities should be focussed on enhancing the life skills like, empathy, interpersonal relationship, communication, critical thinking and creative thinking.

A project report may be prepared and presented by each student after completion of the project. The report may comprise of title of the project, aim of the project, process/es involved, outcomes, limitations and bibliography. Students may refer authentic resources and same may be acknowledged in the report.

Assessment of the project work shall focus on the process involved and its educational value. The honest efforts taken by the students must be acknowledged and due credit may be given, even if the outcomes are not accurate. It may be kept in mind that the aim of project work is to enhance the competencies of the students and not accuracy ore mere completion of a mandatory task.

Introduction to Microsoft Excel

A spreadsheet is also called a worksheet. It is made up of rows and columns that help sort data, arrange data easily, and calculate numerical data. You can also analyse data and depict the same using different plots or graphs. Below is a list of spreadsheet programmes you can use.

- Google sheets (Free and online source)
- Libre office-Calc (Open Source)
- Open office-Calc (Open Source)

Overview

For the purpose of demonstration, MS-Excel functionality is used being most common.These functionalities are equally available in all other applications. Infact, use of free and open source software may be promoted. Spreadsheet is a computer program which is used to create, store, analyzedata in a tabular form i.e.in the form of rows and columns. It performs various mathematical functions, logical and conditional operations in a very quick and accurate manner.Some of its main features include

- Perform Calculations
- Making Charts
- Graphing Tools
- Functions (SUM, AVERAGE, COUNT, etc.)
- Data Analysis
- Business Decision
- Record Expenditures and Income

Get Familiar with Interface

To start with any application you have to open it. Once you open a spreadsheet, you will see the main screen

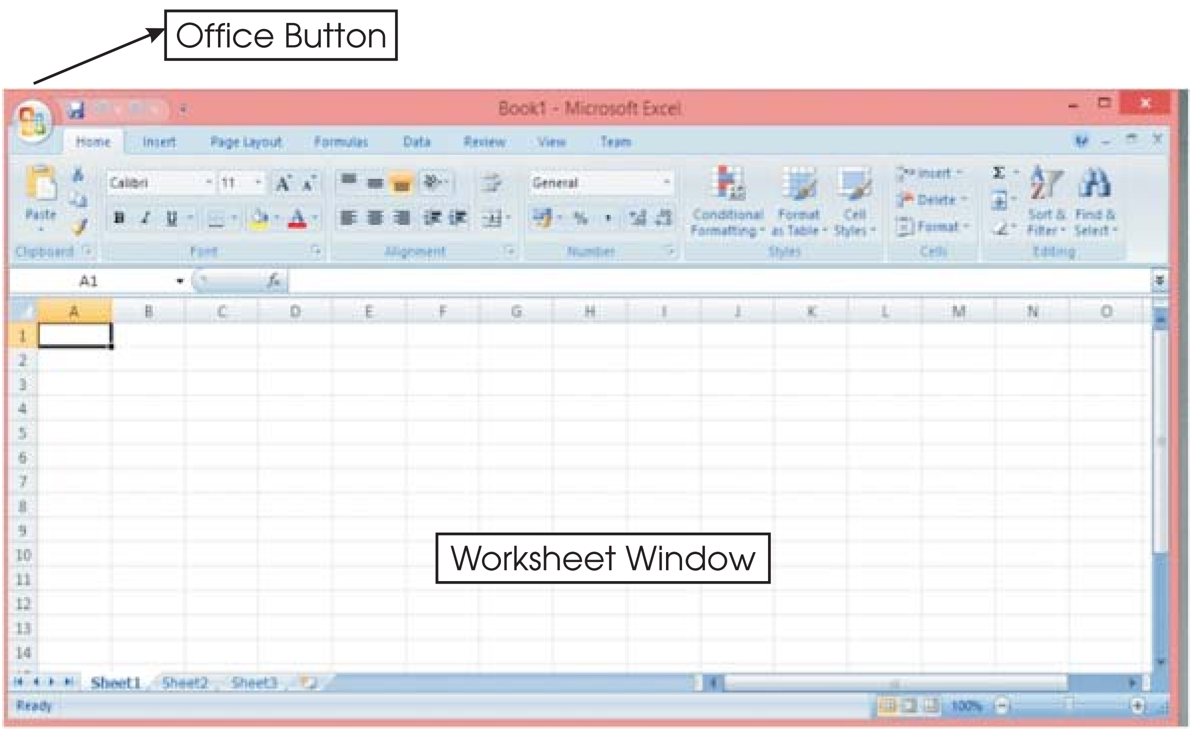


Fig 1: Microsoft Excel Main Screen

The spreadsheet file is called a Workbook and default title is Book1 and by defaultit opens a 3 spreadsheet or worksheets. The tabs at the bottom of the screen represent different worksheets i.e. Sheet1, Sheet2 etc .As you can see the worksheet window consists of several boxes known as cells. Each worksheet contains columns and rows and make like a grid type structure.

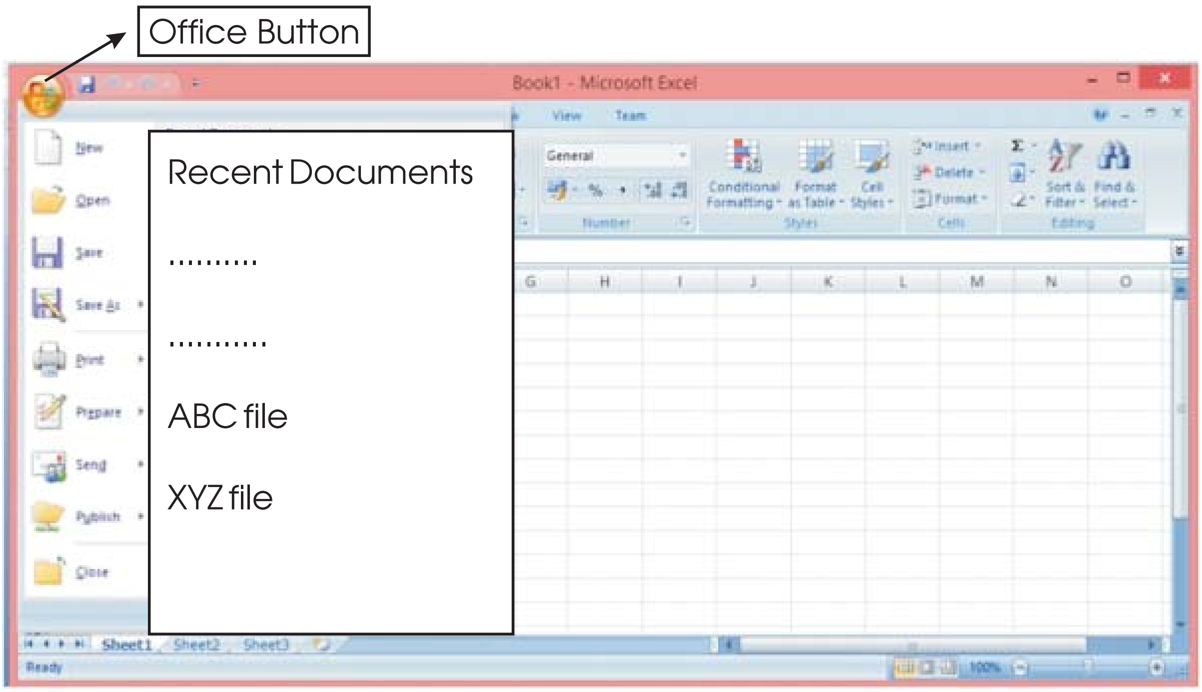


Fig 2: Office Button

Above picture shows Office button contains many options like you'd see in the File menu, such as New, Open, Save, Print, Close etc. Functions of the given menu options are:

1. **New:** To create any new file or new document in the spreadsheet, Word Document etc. Short cut Key to create a new File: **Ctrl+N.**

2 **Open:** To open any existing file from the computer.

3. **Save :** Save the changes to the current open file. If you are working on the existing file and at any point you want to save the file you can use the shortcut **Ctrl+S.**

4. **Save As:** Save a new file with a desired name using the correct file extension and save it to the desired location on the computer's hard drive. UseF12 key for saving any new file.

5. **Print:** It is used to print the hard copy of the document by using printer. You can use shortcut for print **Ctrl+P.**

6. **Close:** To close the current file. Shortcut: **Ctrl+W.** To quit use **Alt F4** on windows.

Interface Elements

1. **Workbook:** It is a collection of one or more worksheets or spreadsheets. Example: Sheet 1 is spreadsheet of workbook Book1.

2. **Ribbon:** It is divided into different Tabs (Home, Insert, Page Layout, Formulas…etc and tabs are divided into groups. Groups are the sets of related to commands that are displayed on tabs.

3. **Name box** (left) and **Formula bar** (**right**)**:** Name Box shows the address of the current cell. Example: Intersection of column and row form a cell, like A1 is located where column A and row 1 meet. Formula bar shows the content of current cell.

4. **Column Headings:** They are denoted by letters and **Row headings** by Number.

5. **Worksheet Navigation Buttons:** It is denoted by Sheet 1, Sheet 2...... And so on. You can use the scrolling buttons on the left to bring other worksheets into view.

6. **Status Bar:** It displays various information like sum, count; average of the currently selected cells. It also displays information about current status of the worksheet etc.

Status Bar Modes

Ready mode. This means nothing is being entered or edited on the spreadsheet.

Enter mode. When you are doing data entry.

Edit mode. Edit the contents of the current cell. Double-click on a cell with data in it, or click inside the formula bar for this mode.

Point mode. Used when linking to cell addresses within a formula or from an spreadsheet.

Dialog window.

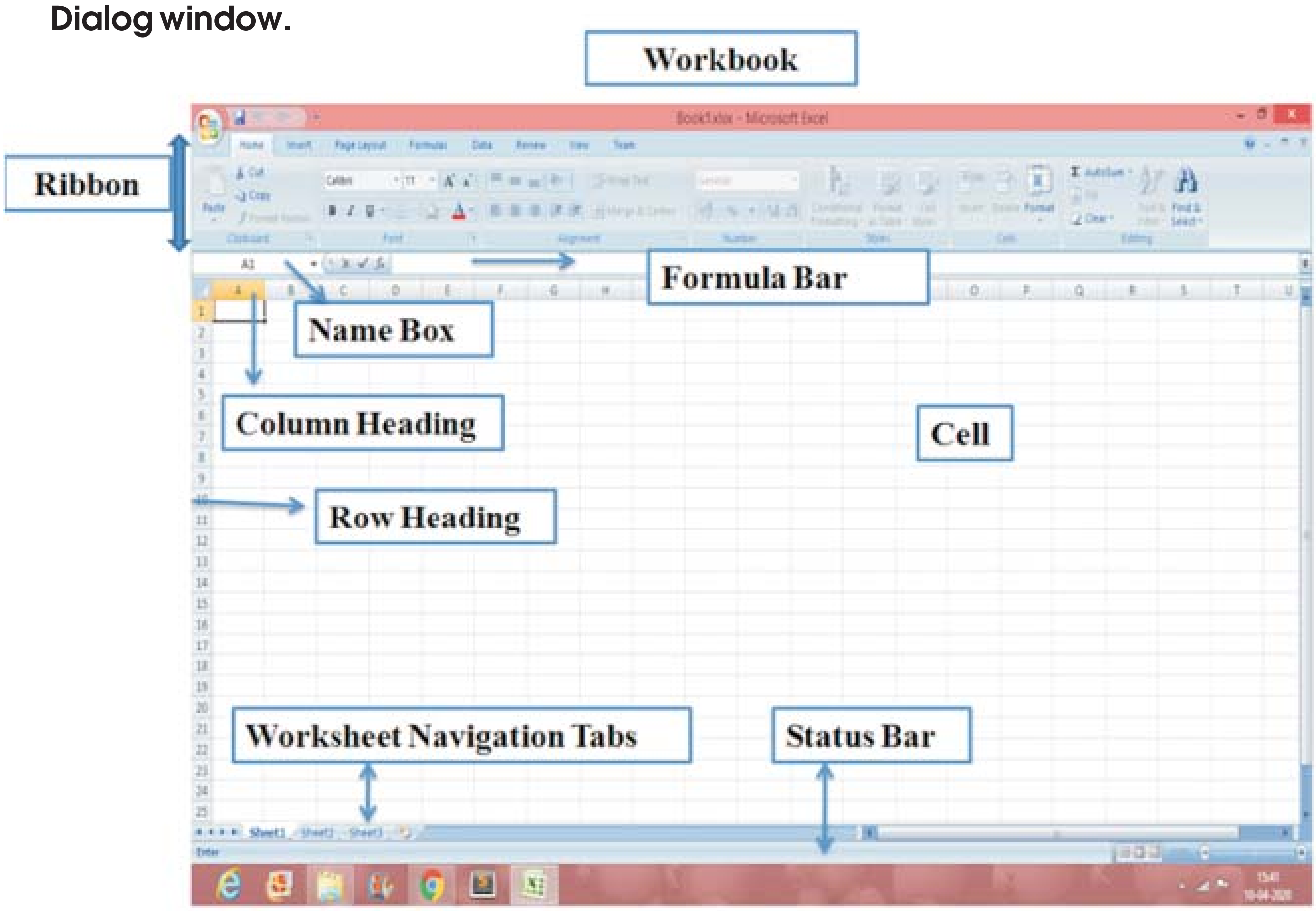


Fig 3: Interface Elements

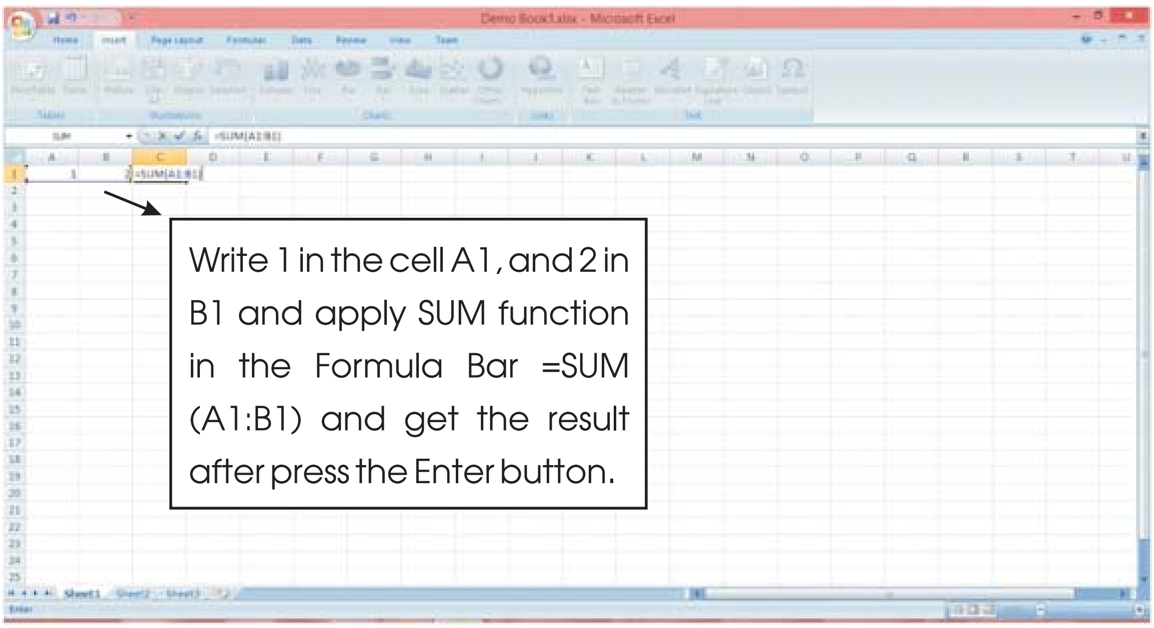


Fig 4: SUM Function

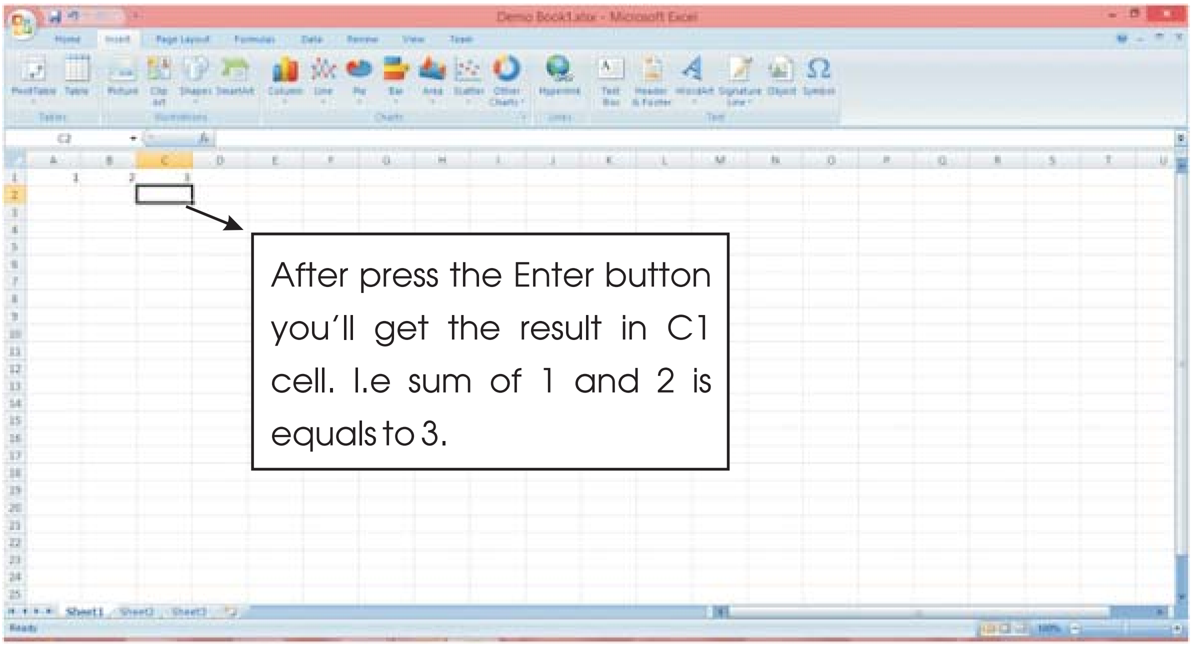


Fig 5: Result of SUM Function

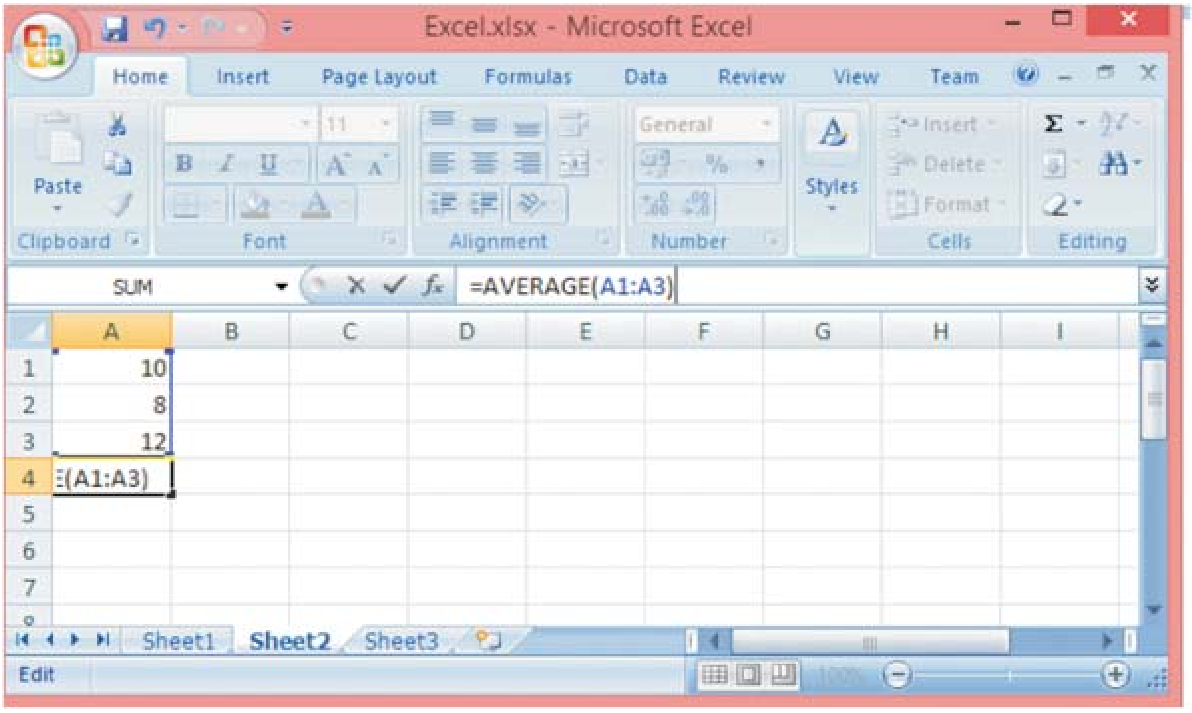
Note: Above is an introduction on the use of spreadsheet. Students should be given practice in sorting and filtering data and use of functions and depicting graphs or plots before starting the practical work.

How to calculate Average in Excel

In Excel, average can be calculated by using built in function "**AVERAGE**"**.** Instead of using "**AVERAGE**" function you can also use **SUM and COUNT** function.

1. **Using AVERAGE Function**

   The AVERAGE function below calculates the average of the numbers in cells A1 through A3.Take any three numbers 10,8,12 and calculate average it gives 10.Select the cell to display the result e.g. cell A4 in the image shown below. Now type "=" sign in the cell or in Formula Bar and type "AVERAGE" then use the parentheses to enclose the cell name ,then press the Enter key to get the result.



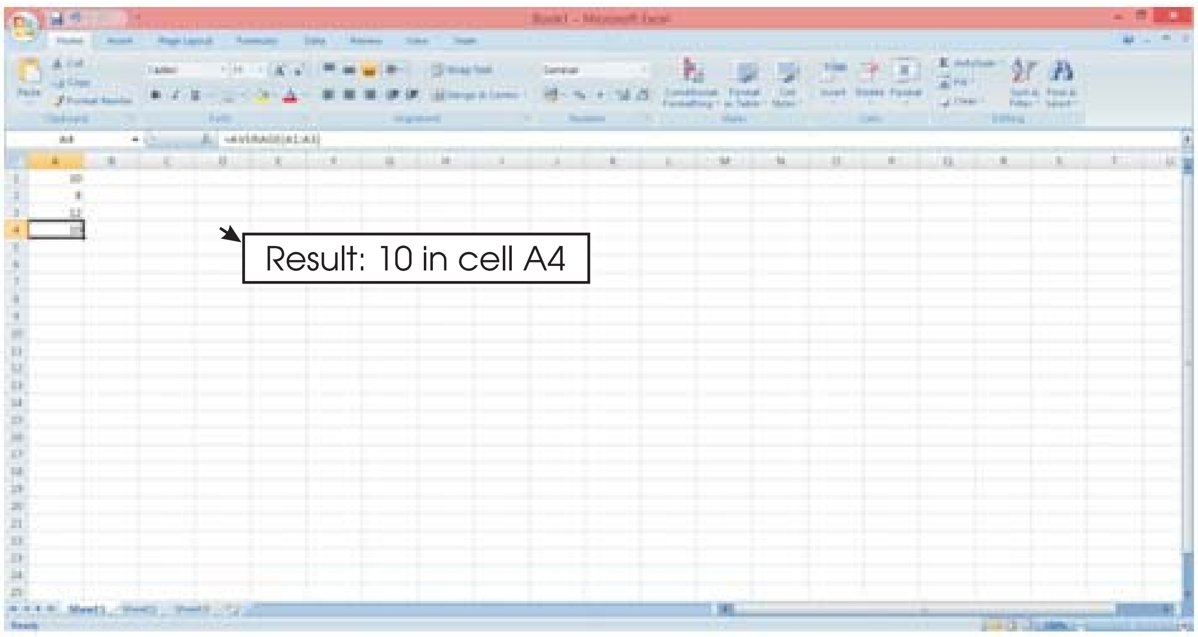


Fig 6: Result of AVERAGE Function

2. **Using SUM and COUNT function:**

The count function in Excel will count the number of cells containing numbers in a given range.It produces the same result i.e 10.

Write the Formula =SUM(A1:A3)/COUNT(A1:A3) in formula bar for calculating the average of three numbers 10,8 and 12 and press enter it will give the result 10 in the cell A4.

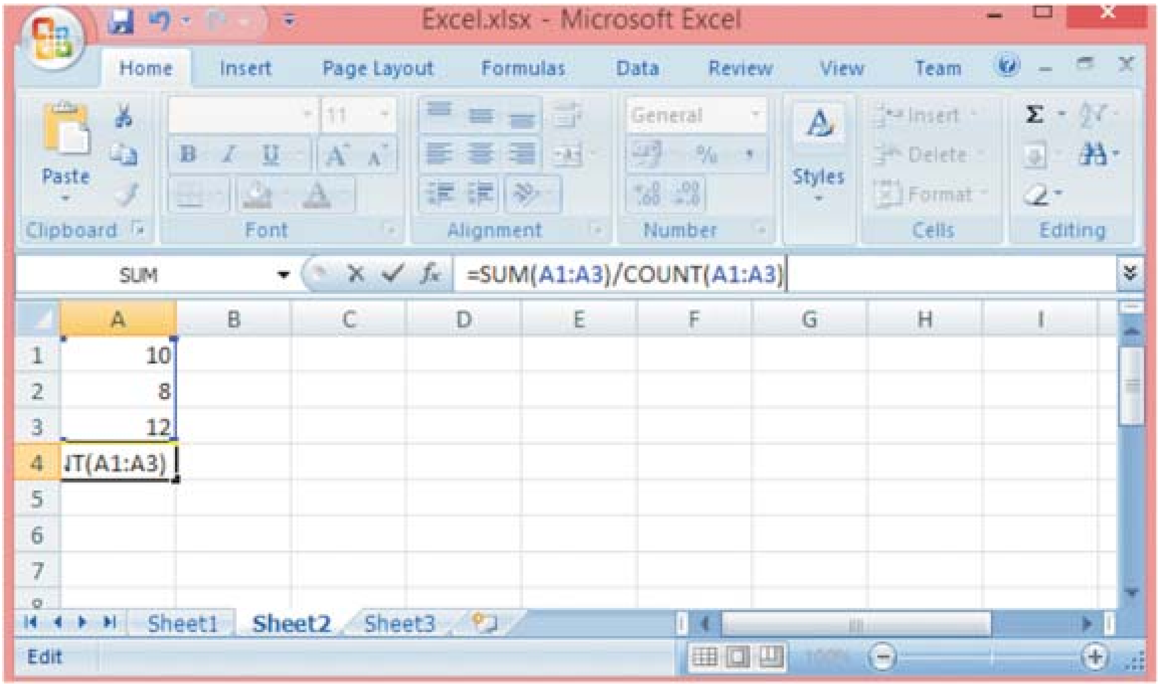


Fig 7: Result of AVERAGE Function using SUM and COUNT function

Let's take more examples to understand the application of using AVERAGE function in Excel. If you want to find the average number of days to complete a tasks by different employees. Or, you want to calculate the average temperature on a particular day over a 10-year time span etc.

Example 1: Suppose you want to calculate the average marks attained by students in all the exams; use the AVERAGE formula to calculate the average marks attained by each student.

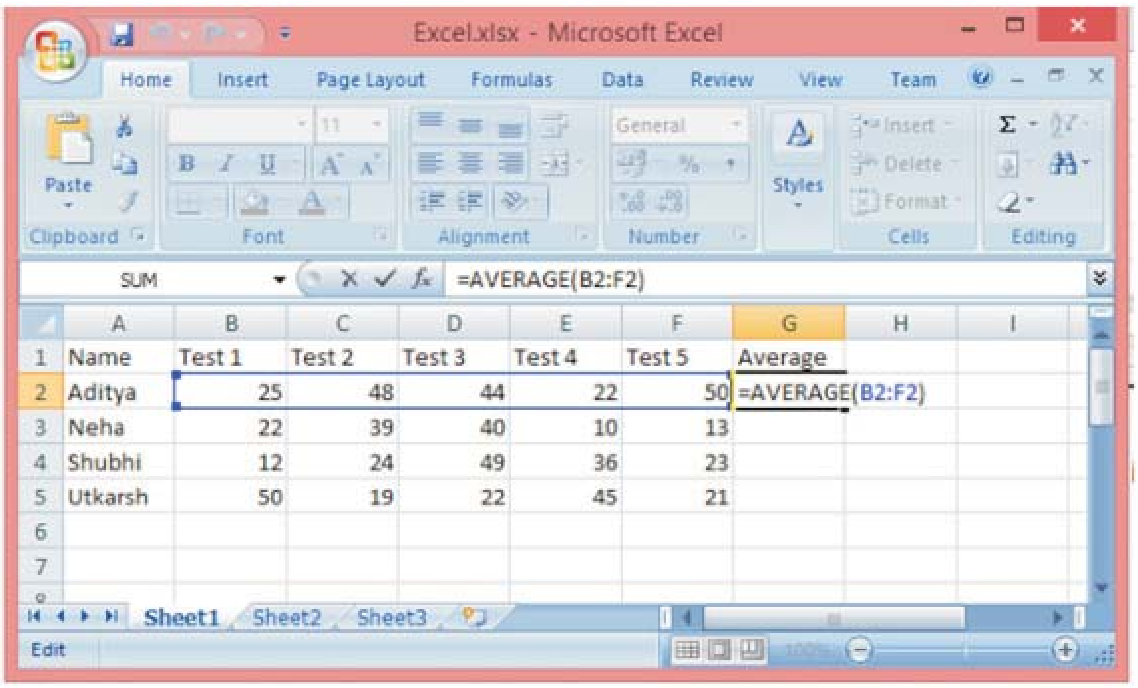


Fig 8: Average calculation of marks

In the above example, I have calculated the average marks for one student in 5 exams.

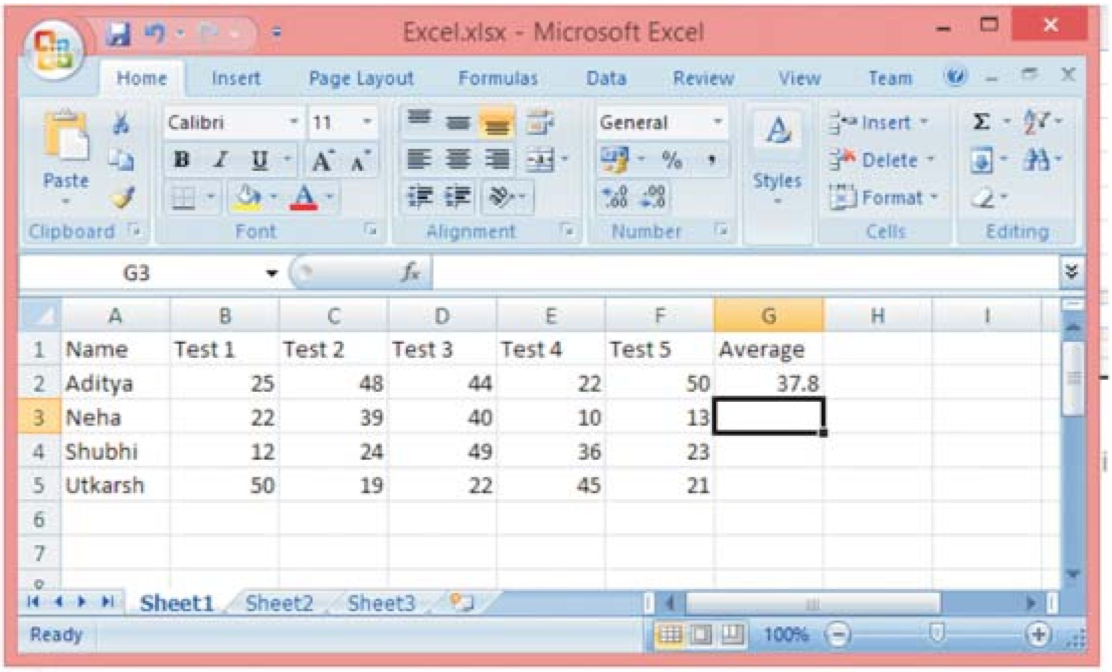


Fig 9: Result

As same as you can calculate the average marks for other students also.

Let's see some other options of Excel AVERAGE Function with examples.

1. **AVERAGEA Function:** The AVERAGE function ignores logical values TRUE or FALSE, empty cells and the cells containing text during calculation. Unlike AVERAGE function, AVERAGEA function will also evaluate the logical values, empty cells, text containing cells.

   The above example ignores the logical values and text during calculation and gives the result 7.

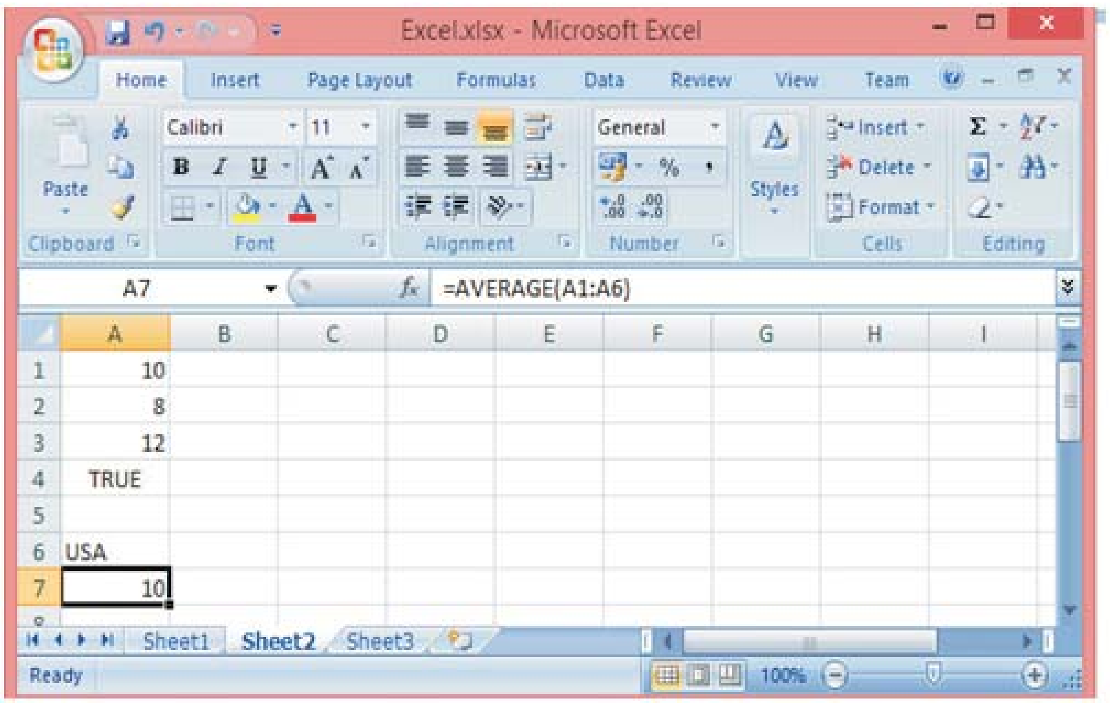


Fig 10:

However, if you using AVERAGEA function the logical value FALSE and cells that contain text evaluate to 0 and the logical value TRUE evaluates to 1. Italso ignores empty cells.

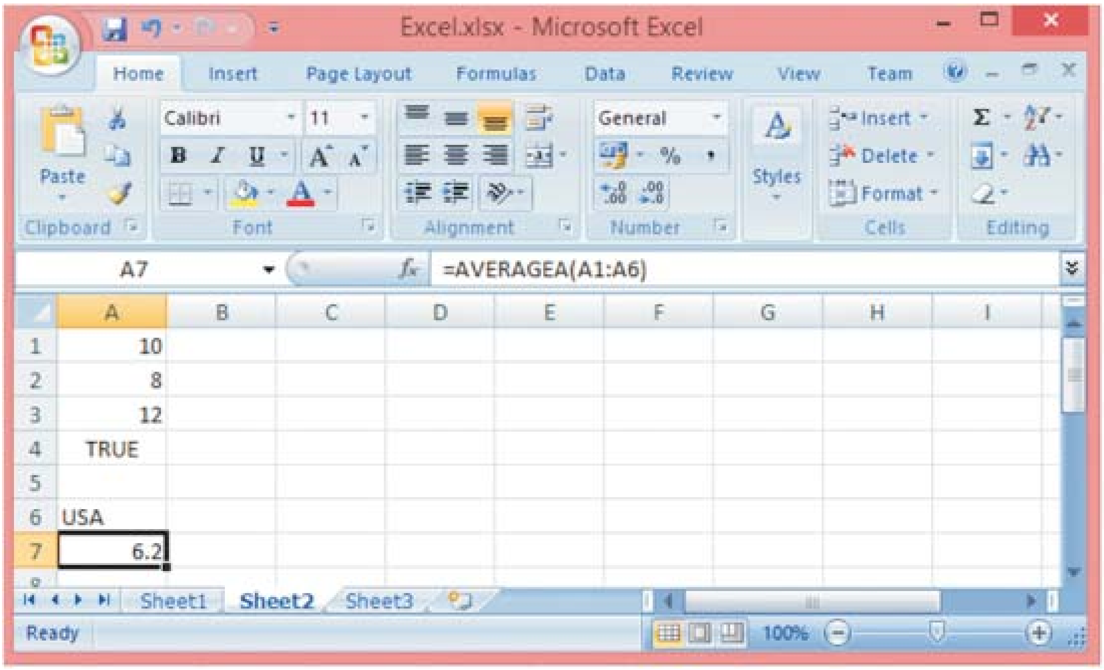


Fig 11: AVERAGEA Function

2. **AVERAGEIF Function:** It calculates the average of cells that meet one criteria. For example, the AVERAGEIF function below (two arguments) calculates the average of all values in the range A1:A7 that are greater than 0.

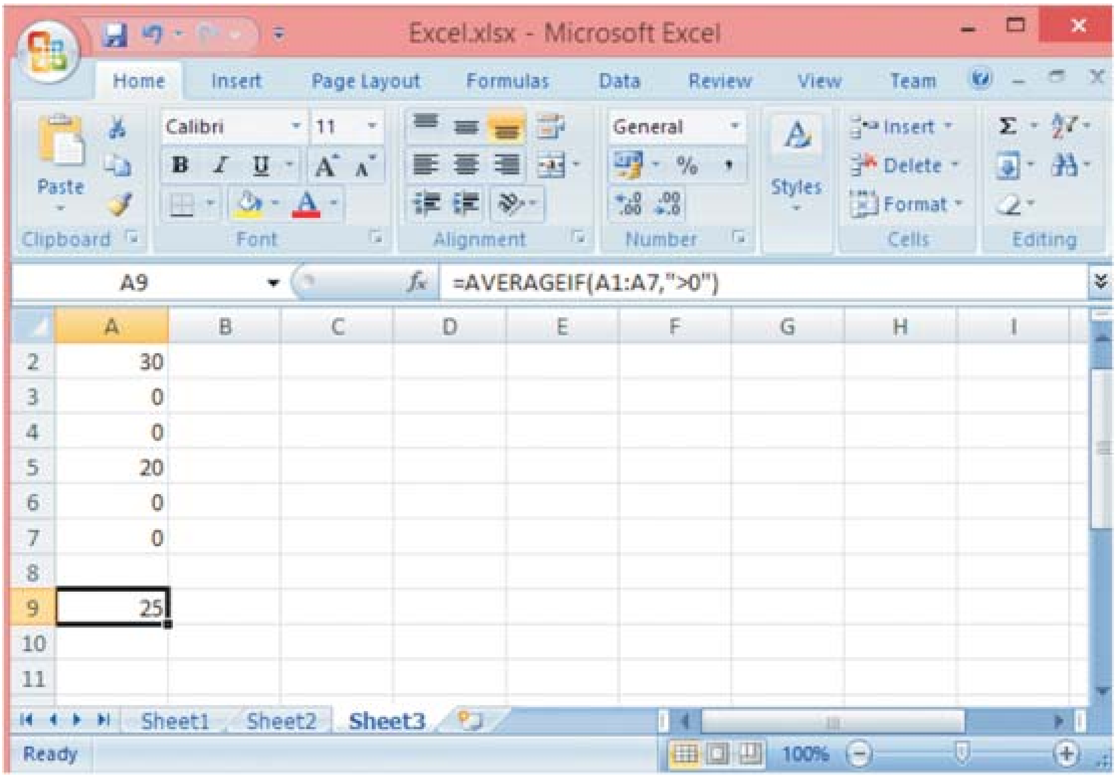


Fig 12: AVERAGEIF Function

Calculate Simple Interest

The general formula for calculating simple interest in Excel is shown below:

Interest = Principal*Rate*Term

Let's take an example to understand that how do we calculate simple interest

Example1: Calculate the simple interest on $3000 invested at the rate of 10% per annum for 2 years.

In the example shown below, so using the cell references we have

B5= B2*B3*B4

= 3000*0.1*2

=600

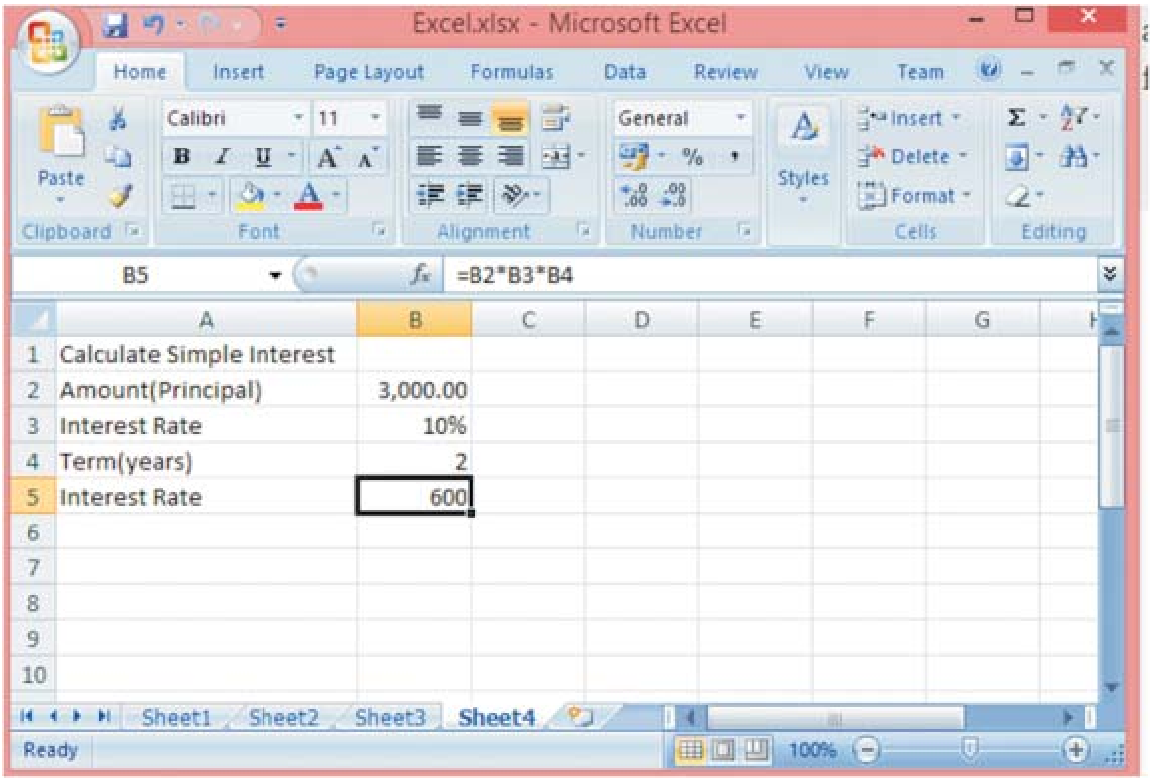


Fig 13: Simple Interest Calculation

Calculate Compound Interest

To calculate compound interest, we can use one of the Excel Formulas called FV. This function will return the future value of an investment on the basis of periodic, constant interest rate and payments. The syntax of this function is as follows:

FV (rate, nper, pmt, pv)

In order to calculate the rate, you will need to divide the annual rate by the number of periods i.e annual rate/ periods. No. of periods or nper is calculated by multiplying the term (no. of years) with the periods i.e term * periods. pmt stands for periodic payment and can be any value including zero.

Let's take an example

Calculate the Compound Interest for $500 at a rate of 10% for 5 years and assuming that the periodic payment value is 0. Please note I have used -B1 meaning, $500 has been taken from me.

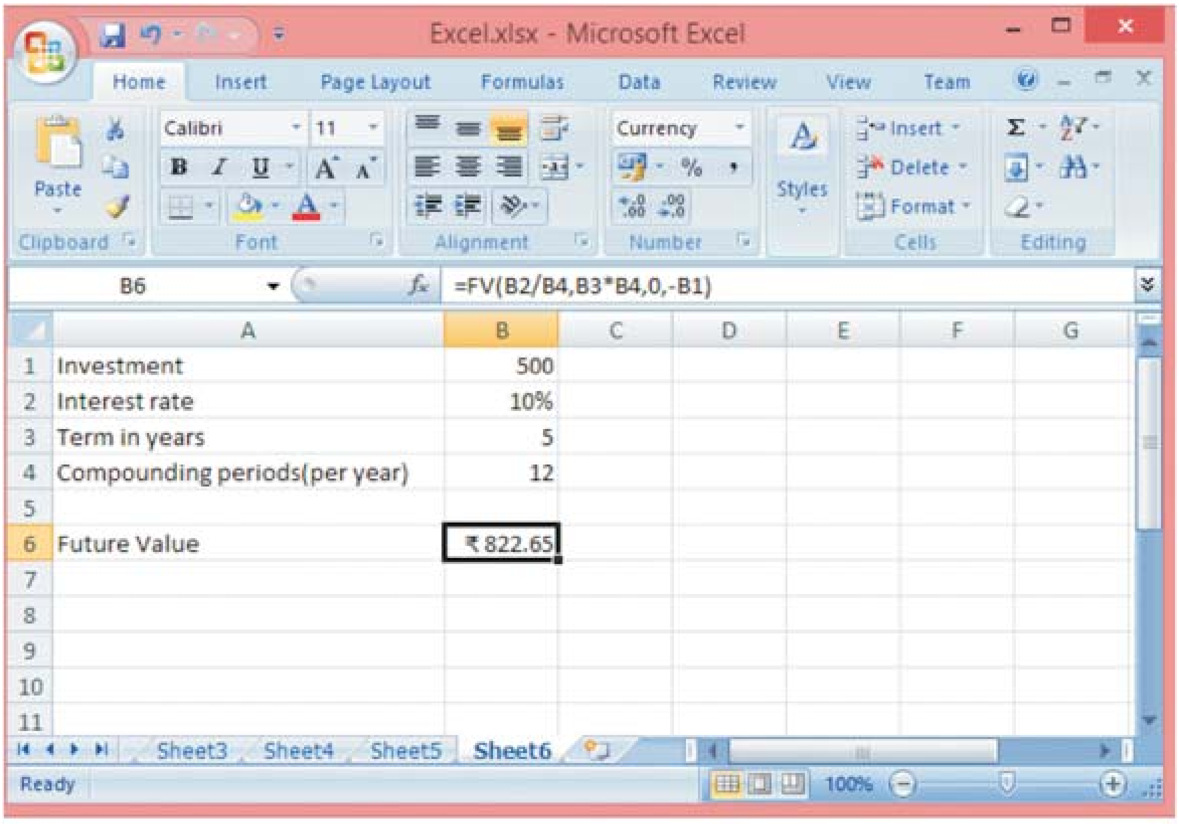


Fig:14 Compound Interest Calculation using FV Function

Note: The Excel FV function is a financial function that returns the future value of an investment. You can use the FV function to get the future value of an investment assuming periodic, constant payments with a constant interest rate.

Pictographs

A pictograph is the presentation of data using images. This is one of the easiest ways to represent statistical data. It helps in representing the large number of data in a simpler form and provides a data visualization by using visual elements like charts, graphs and maps.An example of a pictograph is the cigarette with a red circle and slashes around it means no smoking.

In spreadsheet, pictograph uses images to represent the numerical data in the form of chart or graph. The better we create pictograph, easier would be the visualization and data analysis.

How to create pictograph in Excel

Let's take an example to understand the procedure to create a pictograph .Suppose we have to draw pictograph for favourite fruits of grade 6 students.

I. **To follow this example, open a new worksheet in Excel and enter the following data into the cells referenced.**

- Enter **Apple** into cell **A2.**
- Enter **Banana** into cell **A3.**
- Enter **Orange** into cell **A4.**
- Enter **Grapes** into cell **A5.**
- Enter **Peaches** into cell **A6.**
- Enter **12** into cell **B2.**
- Enter **10** into cell **B3.**
- Enter **8** into cell **B4.**
- Enter **5** into cell **B5.**
- Enter **6** into cell **B6.**

II. **The next step is to create a Bar Graph**

- Drag to select cells **A2 to B6.**
- Select **Insert.**
- Select **Insert Column or Bar Chart** in the **Chart** group.
- Select **2-D Clustered Column.**

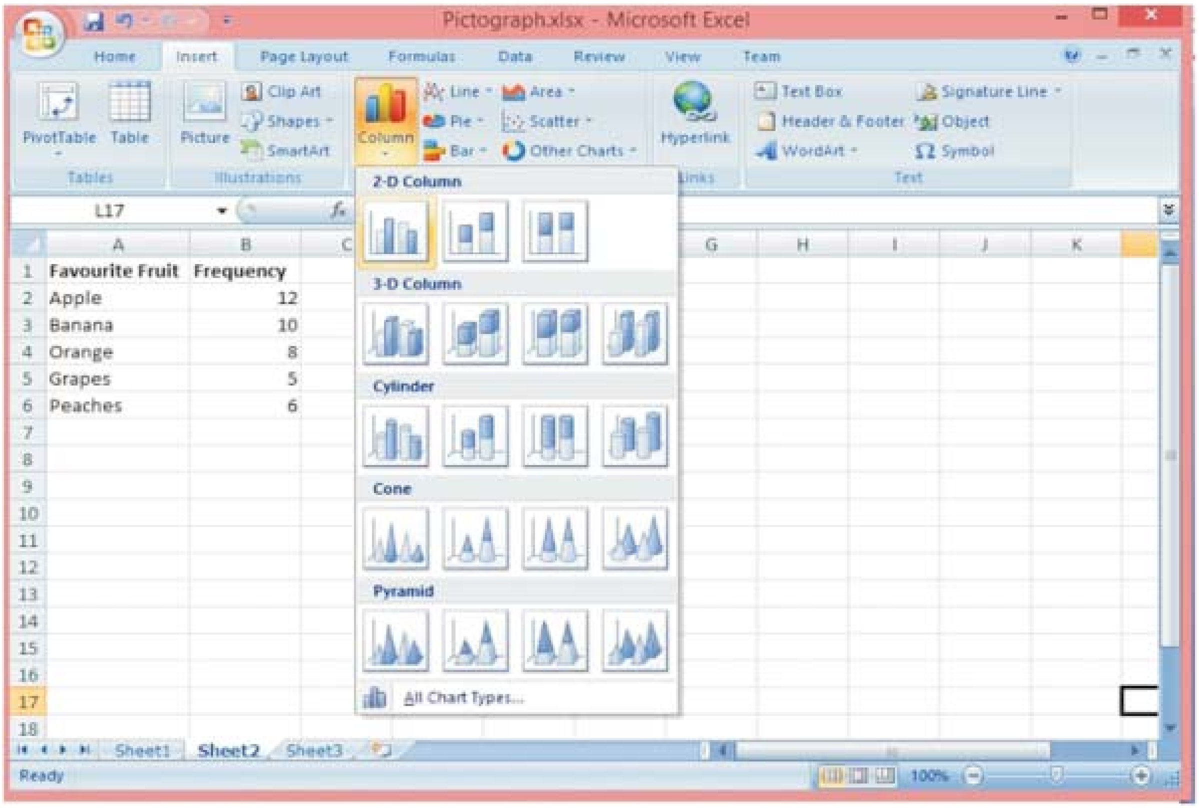


Fig 15: Insert Bar Graph

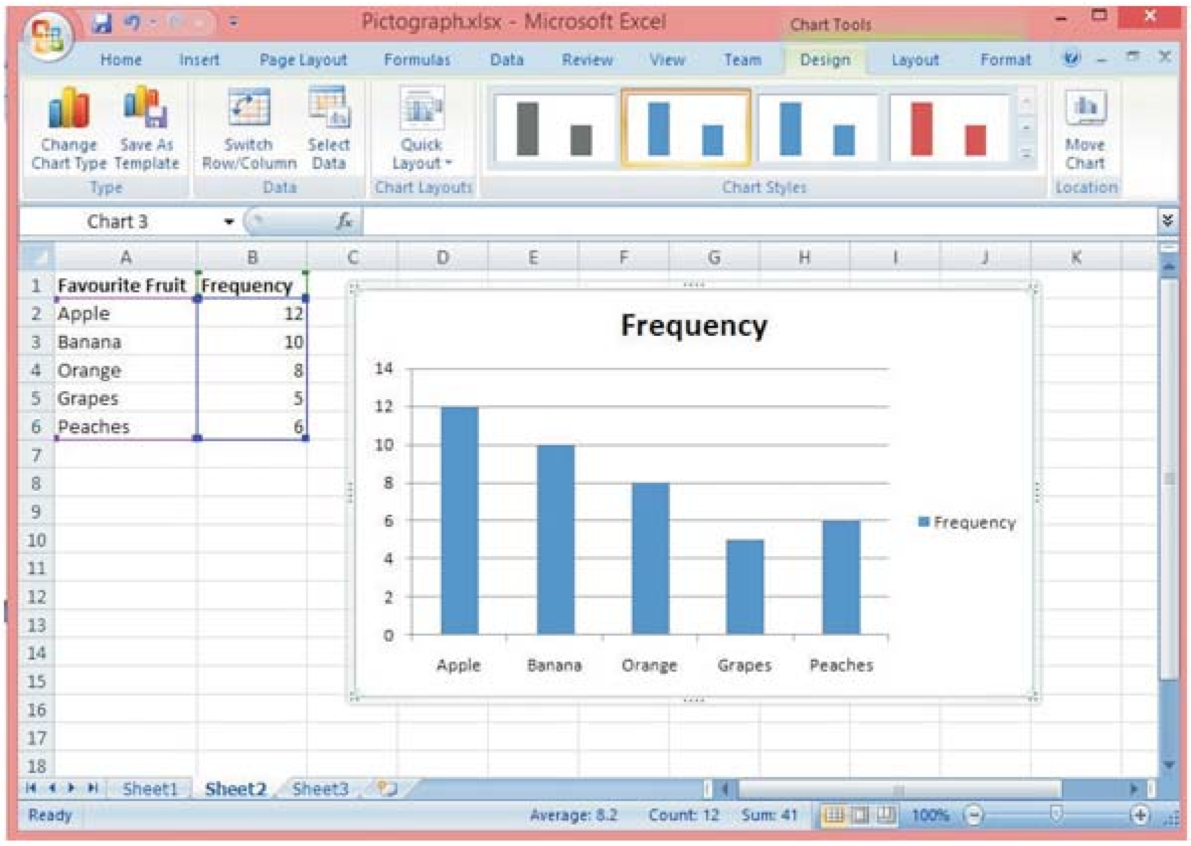


Fig 16: Standard Bar Graph

- A basic column chart is created and placed on your worksheet.
- Add data labels to the graph,select a data bar in the chart right click it and select **Add Data Labels** from the right-clicking menu.

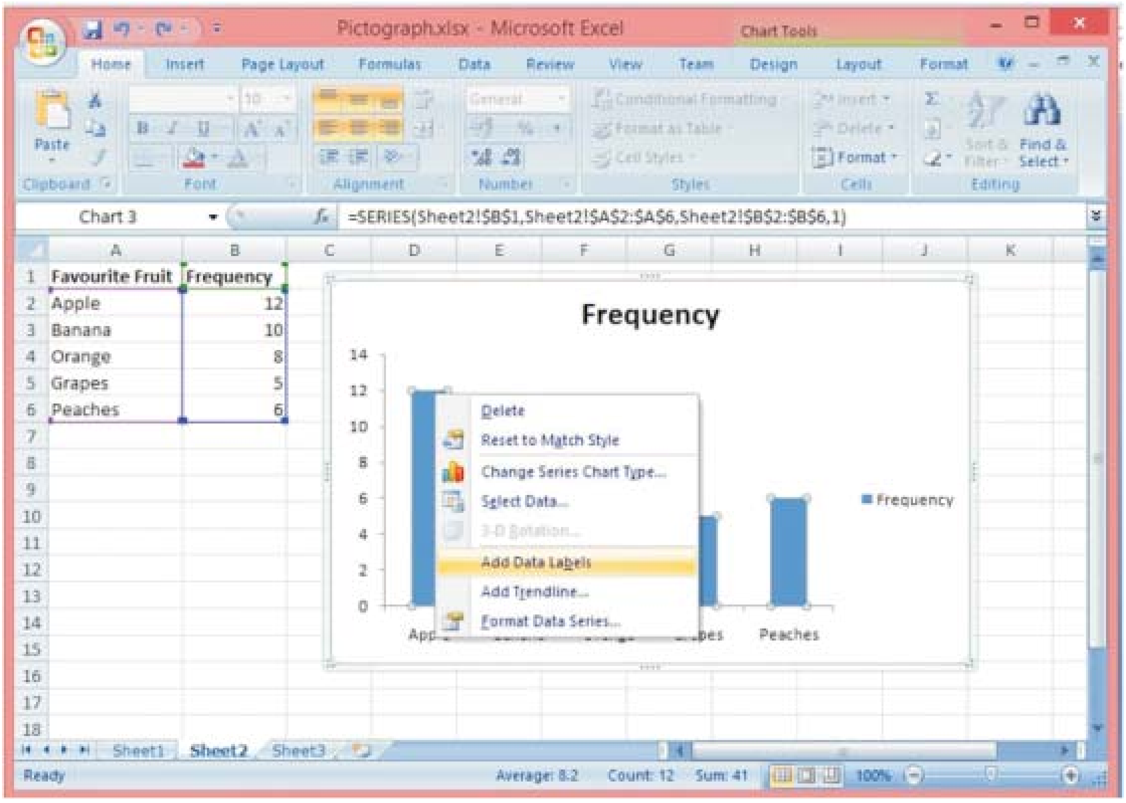


Fig 17: Add Data Labels

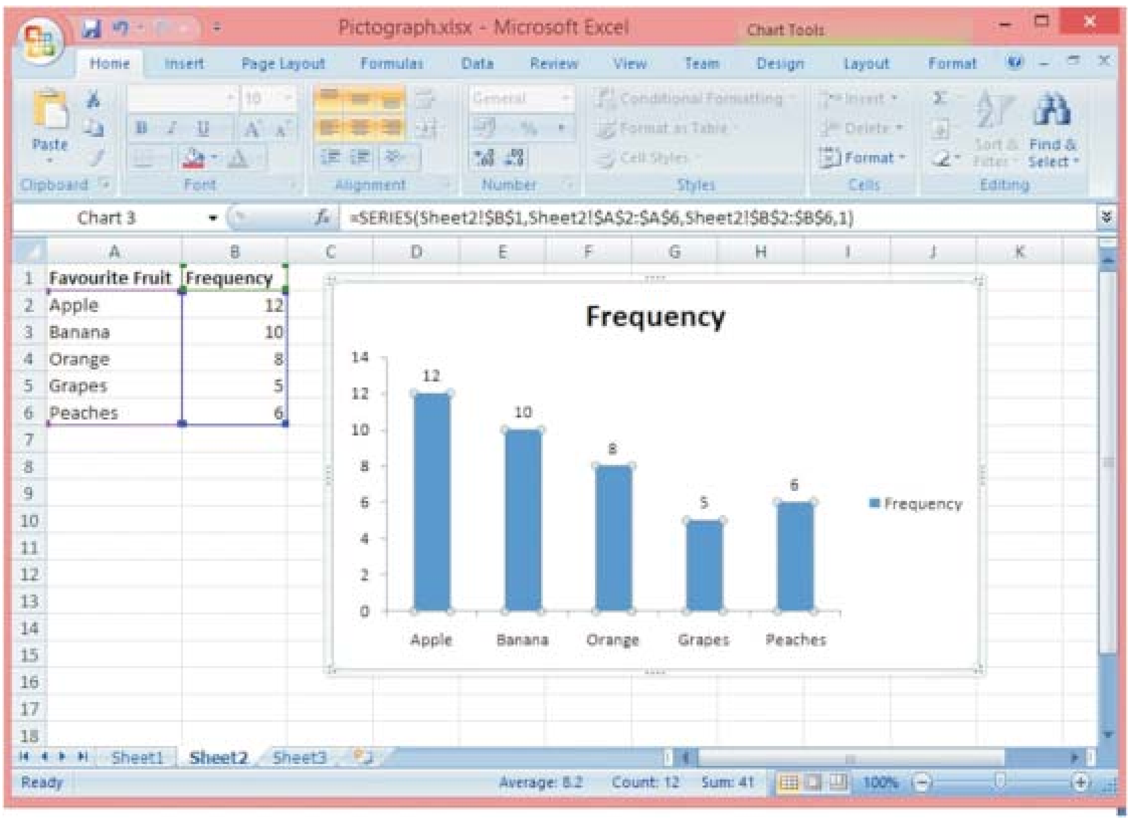


Fig 18: Data Labels in Bar Graph

- Below given figure 18 shows the data labels on the apex of each bar .

III. Thenext step we'll add a picture to the pictograph

- Double-click on one of the blue data bars in the graph and choose **Format Data Series** from the context menu. The **Format Data Series**

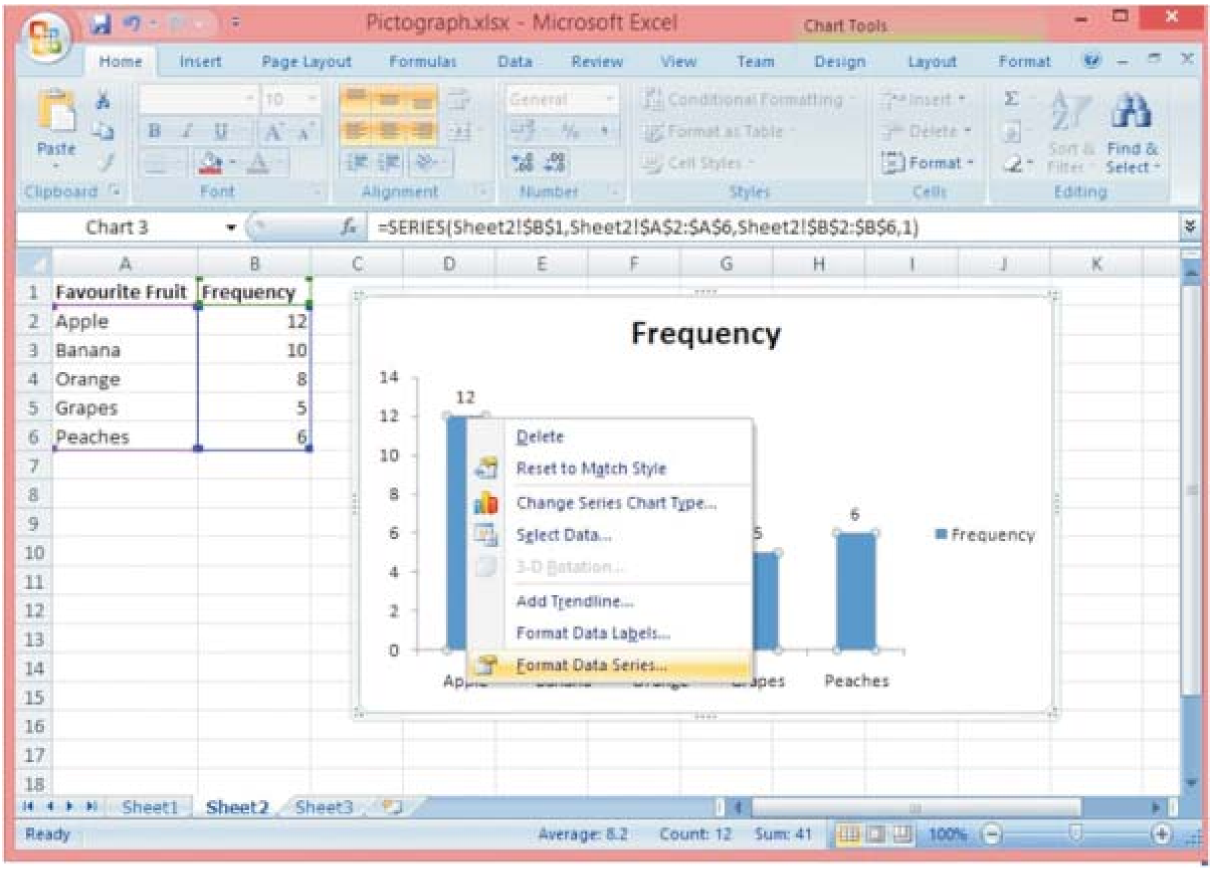


Fig 19: Format Data Series Pane

dialog box opens.

- In the **Format Data Series** pane, you need to select some options as follows

  (i) Select **Fill Options** or the **Fill & Line** icon in the **Format Data Series** dialog box.

  (ii) Select **Picture or texture fill** under **Fill.**

  (iii) Select **File** if you want to use a picture saved on your computer.

  (iv) Select **Online** if you want to search online for a picture to use.

(v) Find and select the image you want to use.

(vi) Select **Insert** to add the picture.

(vii) Click to select the **Stack and Scale with** option in the **Format Data Series** pane.

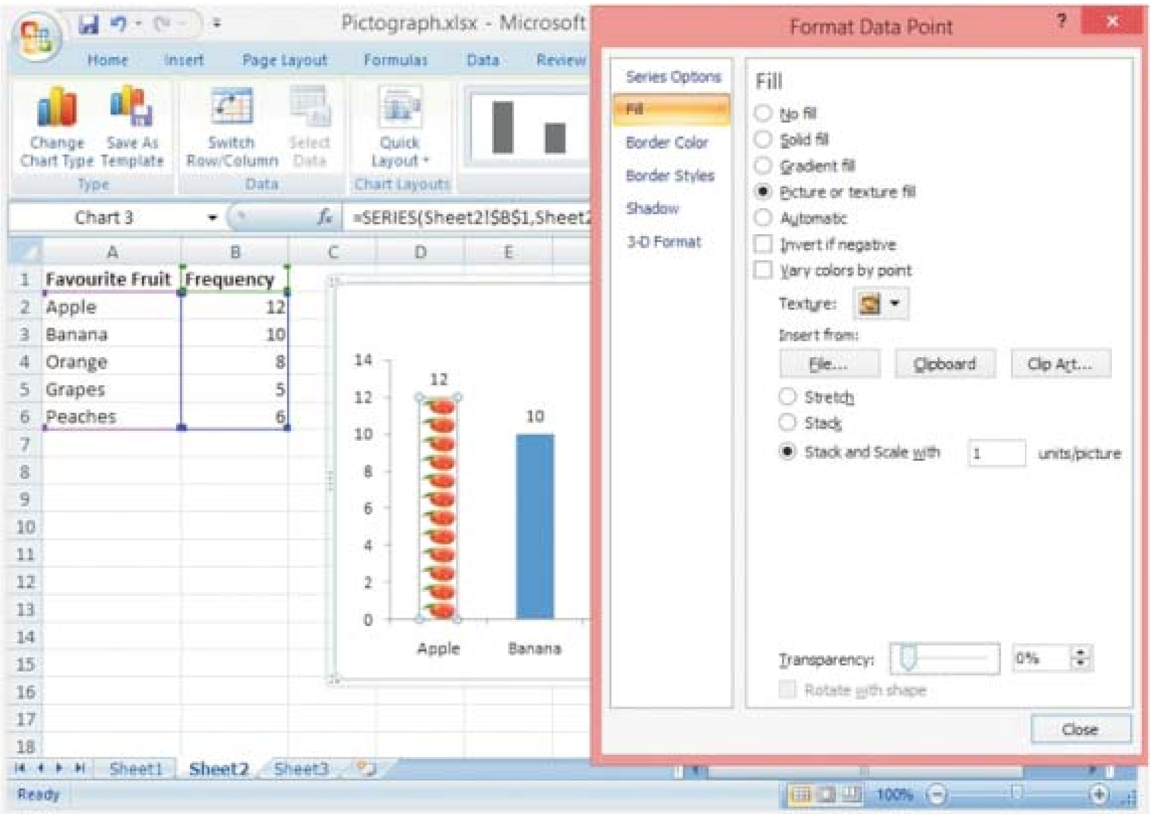


Fig 20: Format Data Points

(viii) Close the **Format Data Series** dialog box. The blue-colored bars in the graph is now replaced with the image selected.

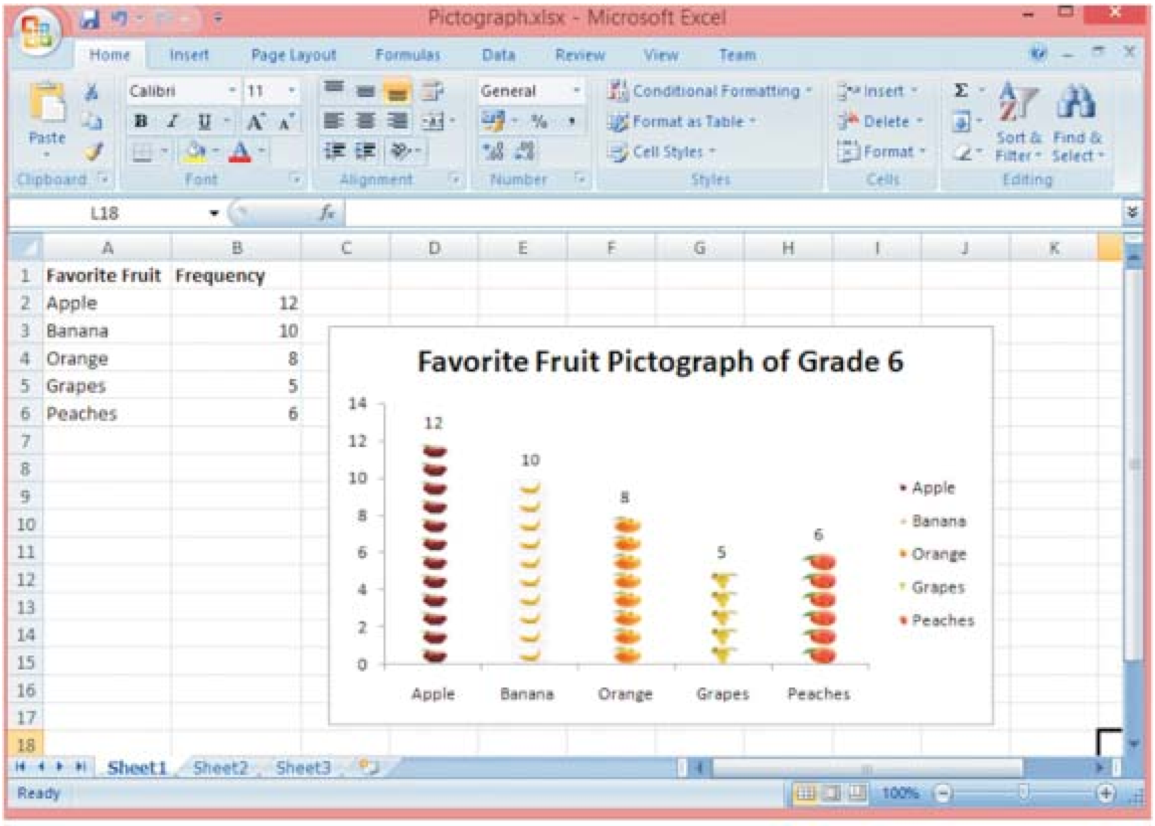
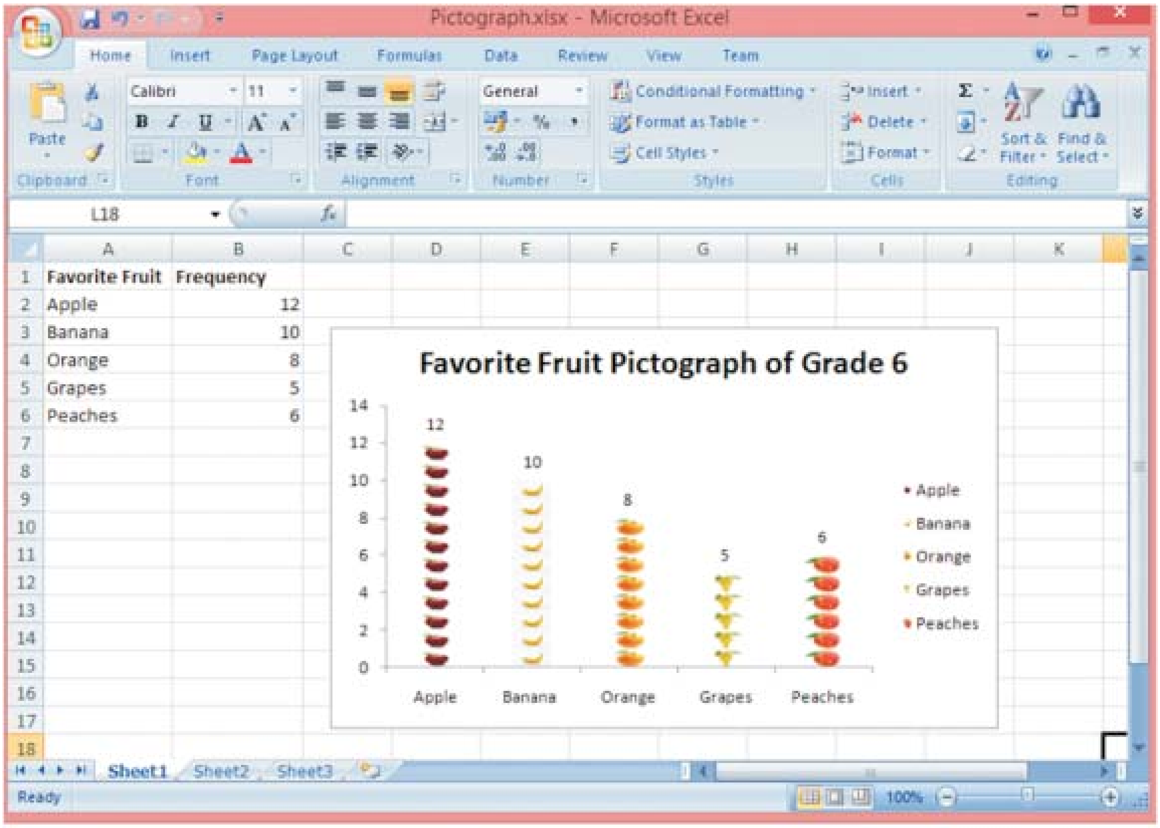


Fig 21: Pictograph

(ix) Repeat the above steps to change the other bars in the graph to pictures.

Create Pie Chart

- **Drag to select cells** A2 to D5.
- **Select** Insert.
- Click on Pie Chart in Chart group.
- Select 2-D Pie
- A basic Pie chart is created and placed on your worksheet.



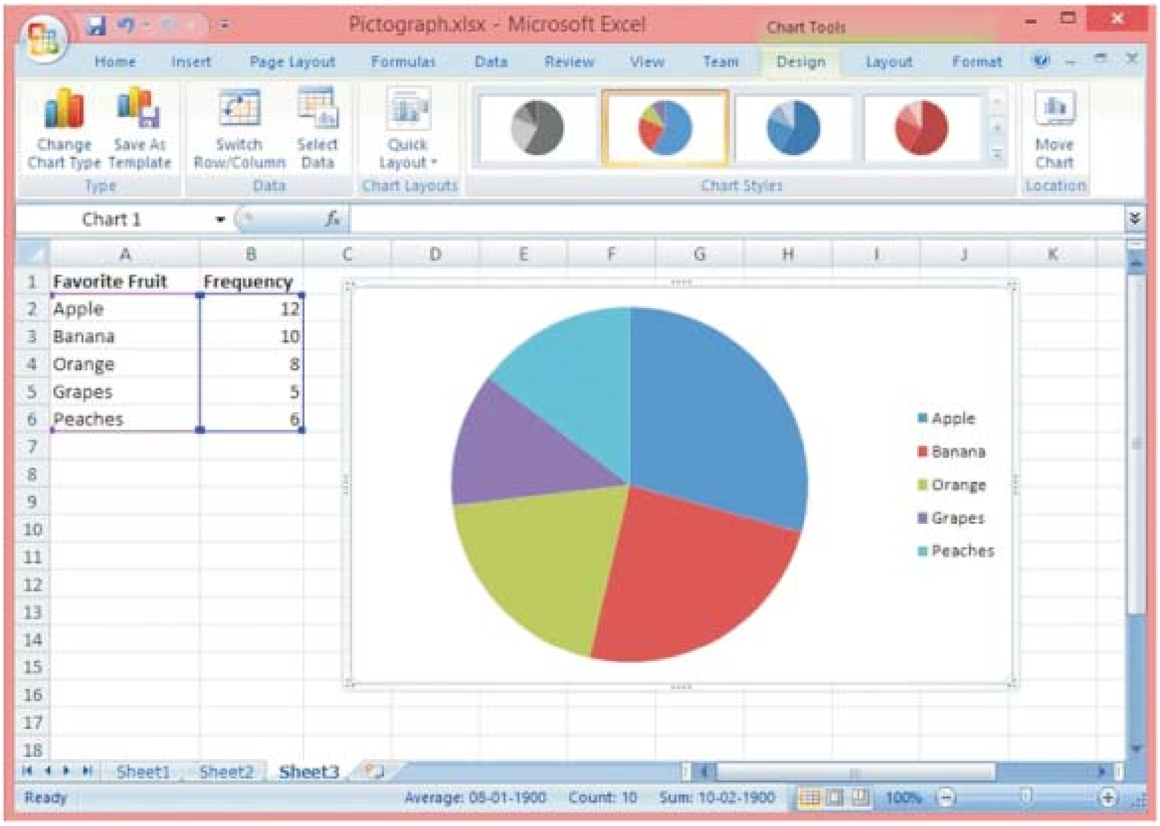


Fig 22: Pie Chart

Calculate Central Tendency

There are three main measures of central tendency: Mean, Median and Mode

Arithmetic Mean : It is also referred to as average. The mean is calculated by adding up a group of numbers and then dividing the sum by the count of those numbers.

In Microsoft Excel, the mean can be calculated by using one of the following functions: AVERAGE, AVERAGEA, AVERAGEIFS, and AVERAGEIF.

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