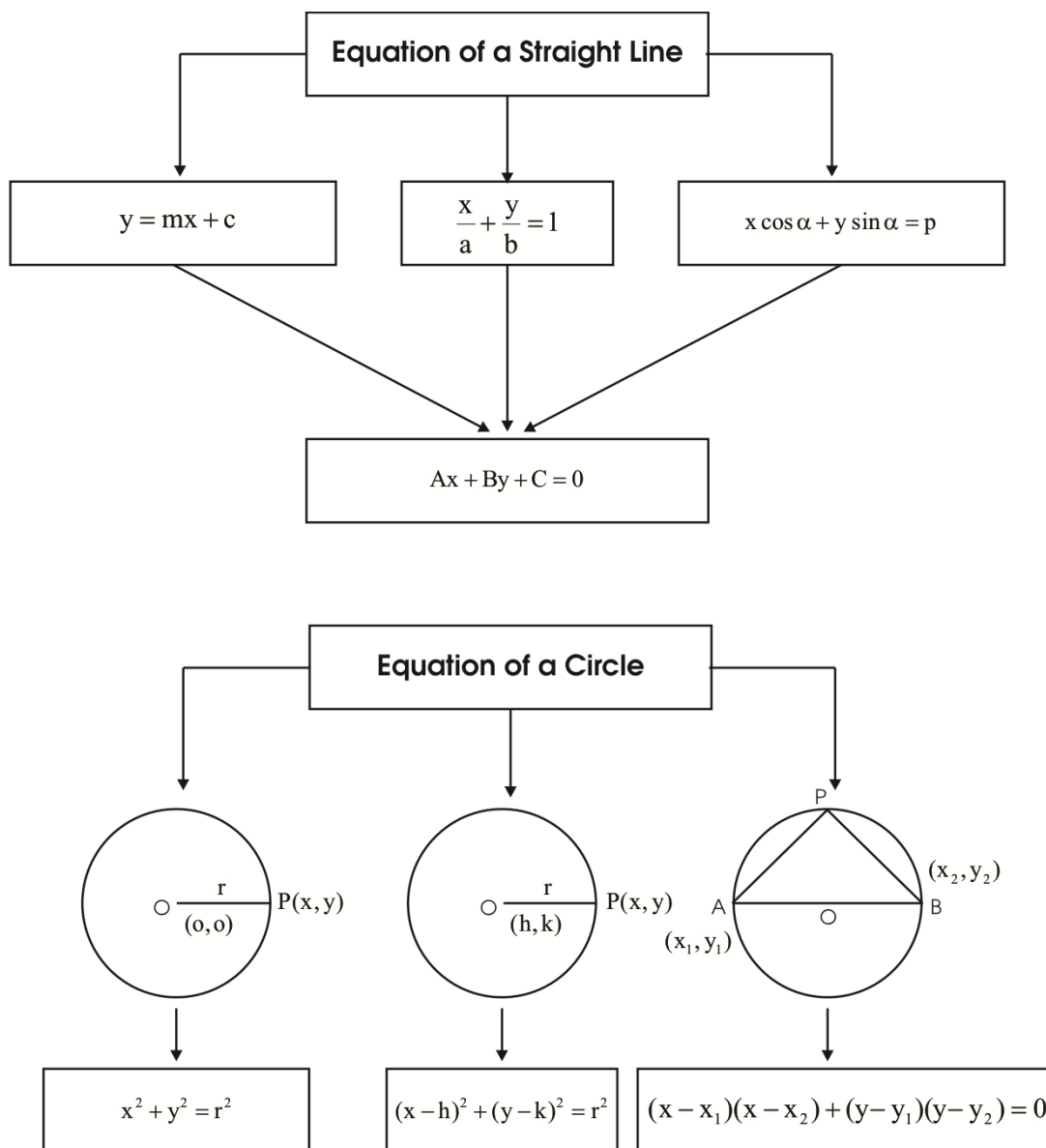# Chapter-XII

Coordinate Geometry

Learning Objectives:

Students will be able to:

- Form equation of a straight line in different forms.
- Form general equation of a line.
- Reduce the general equation of a line into different forms.
- Find the distance of a point from a line.
- Form equation of a circle in different forms.
- Write the general form of the equation of a circle.
- Form equation of a parabola in standard form.

Concept Map

```
                    Equation of a Straight Line
```

$$y = mx + c$$

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$x \cos\alpha + y \sin\alpha = p$$

$$Ax + By + C = 0$$

```
                    Equation of a Circle
```

$$r \quad P(x, y) \quad (o, o)$$

$$r \quad P(x, y) \quad (h, k)$$

$$P \quad (x_2, y_2) \quad A \quad B \quad (x_1, y_1)$$

$$x^2 + y^2 = r^2$$

$$(x - h)^2 + (y - k)^2 = r^2$$
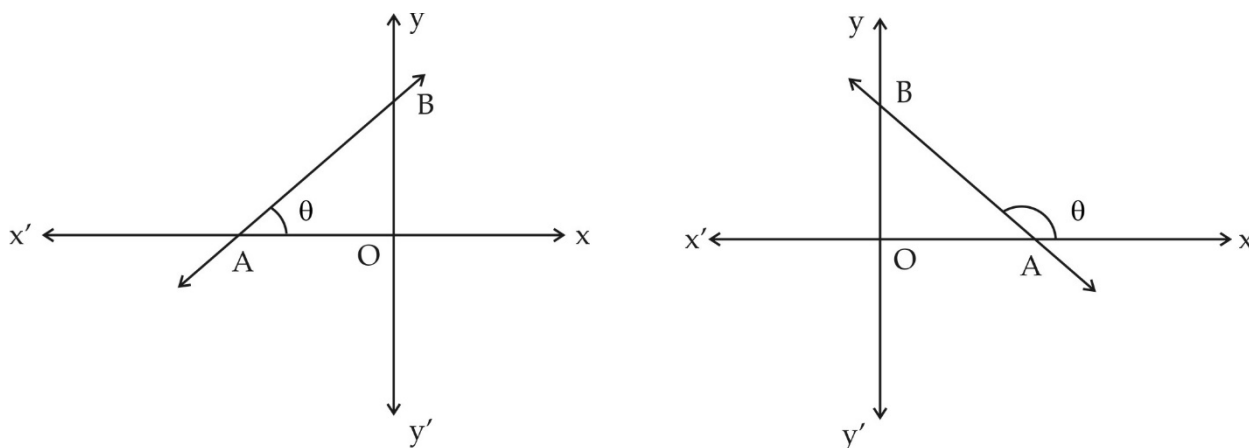
$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

12.1 Straight Line

A straight line is a curve such that every point on the line segment joining any two points on it lies on it.

Equation of a Line: The equation of a straight line is a linear relation between two variables x and y, which is satisfied by the coordinates of each and every point on the line and is not satisfied by any other point in the Cartesian plane.

Slope of a Line:

Inclination of a Line

The inclination of a line is the angle θ which the line makes with the positive direction of the x-axis, measured in anticlockwise direction. Clearly $0^0 \le \theta \le 180^0$. If we are keeping θ unique thus $0^0 \le \theta < 180^0$.

Slope or Gradient of a Line: If θ is the inclination of a non-vertical line, then $\tan \theta$ is called the slope of the line and is denoted by m.

Remark 1: The slope of a horizontal line is 0, because $\tan 0^0 = 0$.

Remark 2: The slope of a vertical line is not defined, because $\tan 90^0$ is not defined.

Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2 respectively. Let their inclinations be θ_1 and θ_2 respectively.
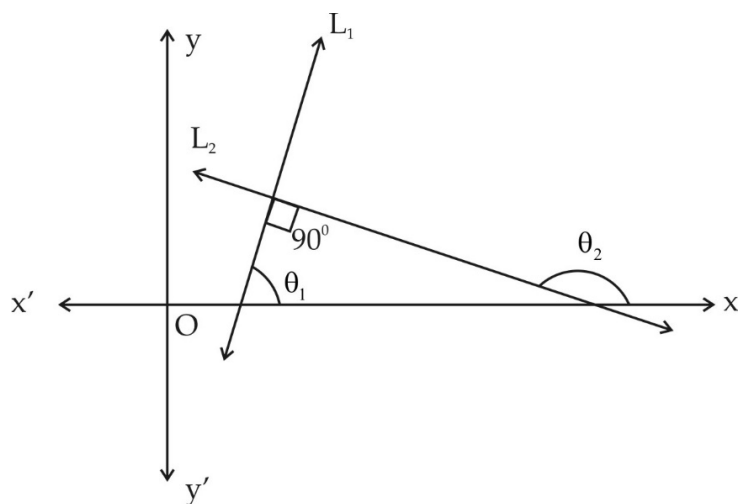
If L_1 and L_2 are parallel

$$\Rightarrow \theta_1 = \theta_2$$

$$\Rightarrow \tan \theta_1 = \tan \theta_2$$

$$\Rightarrow m_1 = m_2$$

(3)

12.2 Slopes of Perpendicular Lines

Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2 respectively. Let their inclinations be θ_1 and θ_2 respectively. Then $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$
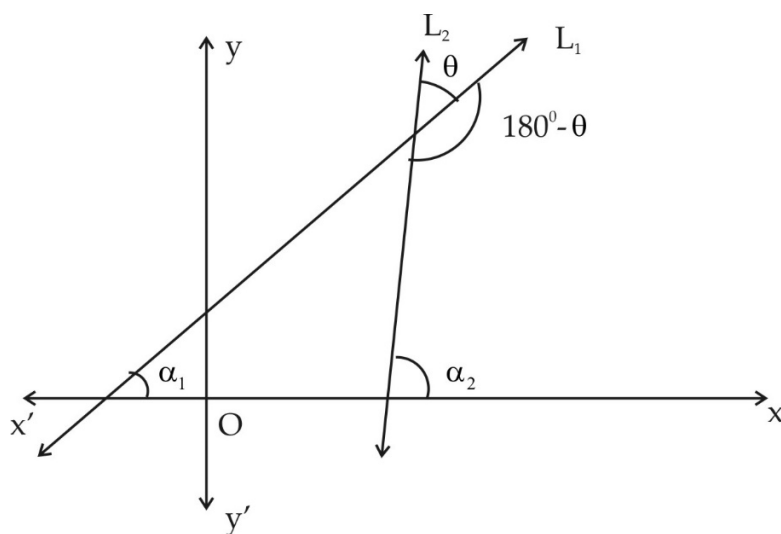
Let L_1 and L_2 be perpendicular to each other. Then $\theta_2 = 90^0 + \theta_1$

$$\Rightarrow \tan\theta_2 = \tan(90^0 + \theta_1) = -\cot\theta_1$$
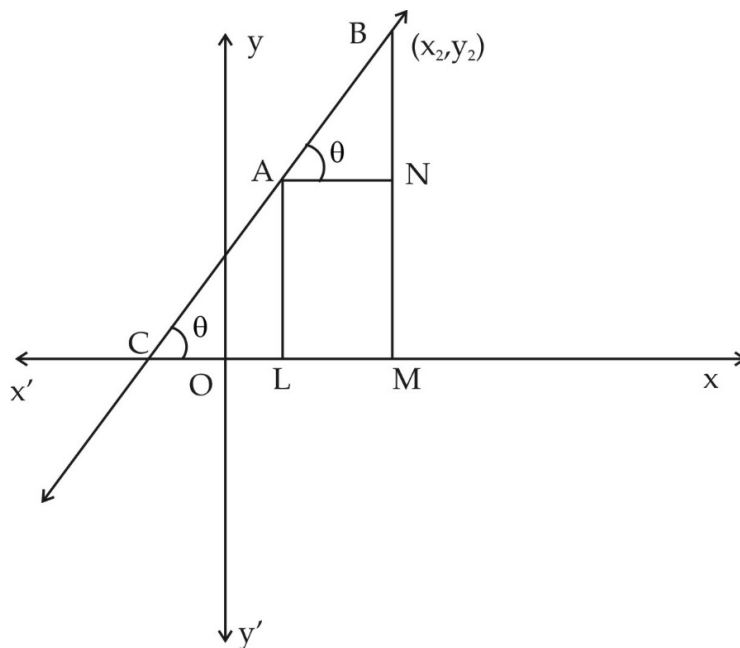
$$\Rightarrow \tan\theta_2 = -\frac{1}{\tan\theta_1}$$

$$\Rightarrow m_2 = -\frac{1}{m_1} \Rightarrow m_1 m_2 = -1$$

Angle between Two Non-vertical and Non-perpendicular Lines

Slope of a Line Passing through Two given Points

Let us consider a non-vertical line CAB, passing through the points A (x_1, y_1) and B (x_2, y_2). Let θ be its inclination Draw AL and BM perpendicular to x-axis and AN $\perp$ BM

Thus $\angle BAN = \angle ACM = \theta$

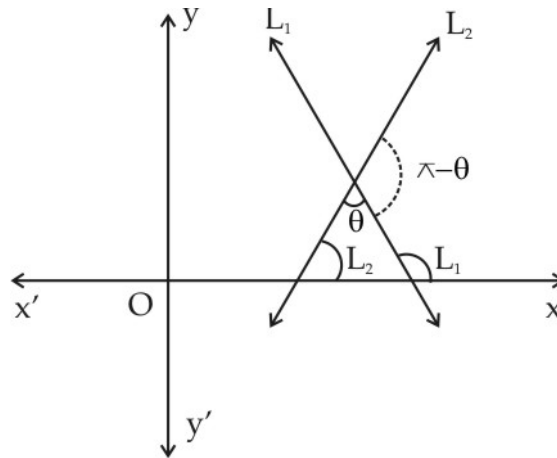From right triangle ANB, we have

$$\tan \theta = \frac{BN}{AN} = \frac{BM - NM}{LM} = \frac{BM - AL}{OM - OL} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow m = \tan \theta = \frac{y_2 - y_1}{x_2 - x_1}$$

Hence, the slope of the line joining the points A (x_1, y_1) and B (x_2, y_2) is given by,

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

12.3 Angle between two Lines

Let L_1 and L_2 be the two given non-vertical and non-perpendicular lines with slopes m_1 and m_2 respectively. Let α_1 and α_2 be their respective inclinations. Then $m_1 = \tan \alpha_1$ and $m_2 = \tan \alpha_2$.

Let θ and $(180^0 - \theta)$ be the angles between L_1 and L_2.

We have $\theta = \alpha_2 - \alpha_1$ where $\alpha_1, \alpha_2 \neq 90^0$

$$\Rightarrow \tan \theta = \tan (\alpha_2 - \alpha_1) = \frac{\tan \alpha_2 - \tan \alpha_1}{1 + \tan \alpha_1 \times \tan \alpha_2}$$

$$\Rightarrow \tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2} \qquad \text{where } 1 + m_1 m_2 \neq 0$$

Also $\tan (180^0 - \theta) = - \tan \theta = - \dfrac{m_2 - m_1}{1 + m_1 m_2}$

Thus, tangent of the angle between given lines is

$$= \pm \frac{m_2 - m_1}{1 + m_1 m_2}$$

Let $\alpha = \min. \{ \theta, (180 - \theta)\}$, then α is acute and so $\tan \alpha > 0$.

$$\therefore \ \ \tan \alpha = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|$$
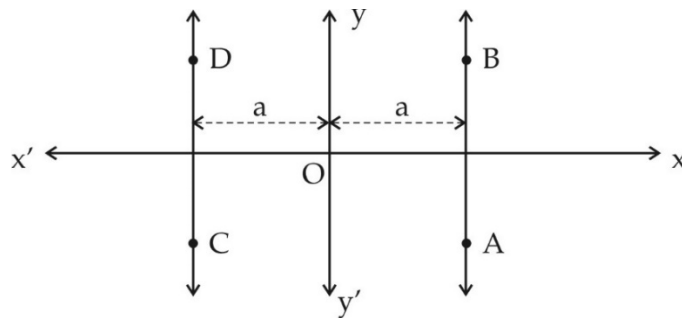
12.4 Various Forms of the Equation of a Line

1. **Equation of x-axis:** We know that the ordinate of each point on the x-axis is 0. If P (x, y) is any point on the x-axis, then y = 0.

   Hence the equation of x-axis is y = 0.

2. **Equation of y-axis:** We know that the abscissa of each point on the y-axis is 0. If P (x, y) is any point on the y-axis, then x = 0.

   Hence the equation of y-axis is x = 0.

3. **Equation of a line parallel to y-axis.**
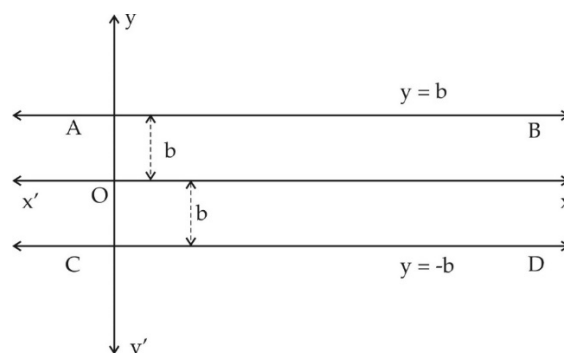

   Let AB be a straight line parallel to the y-axis lying on its right hand side at a distance a from it. Thus, the abscissa of each point on AB is a.

   If P (x, y) is any point on AB, then x = a. Thus, the equation of a vertical line at a distance a from the y-axis, lying on its right hand side is x = a.

   Similarly, the equation of a vertical line at a distance a from the y-axis, lying on its left-hand side is x = –a.
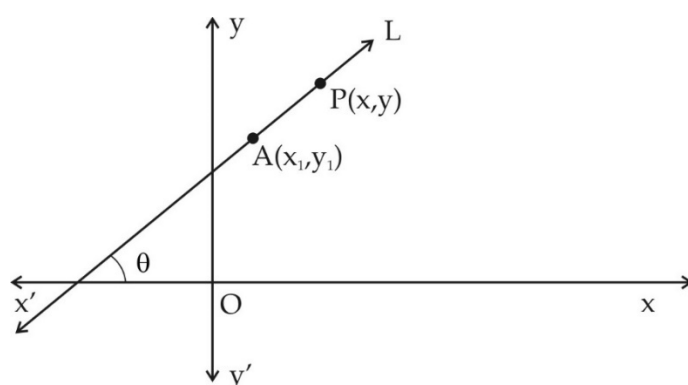
4. **Equation of a line parallel to x-axis**

Let AB be a straight line parallel to the x-axis, lying above it at a distance b from it. Then ordinate of each point on AB is b. If P (x, y) is any point on AB, then y = b. Thus the equation of a horizontal line at a distance b from the x-axis, lying above the x-axis is y = b.

Similarly, the equation of a horizontal line at a distance b from the x-axis, lying below the x-axis is y = –b.
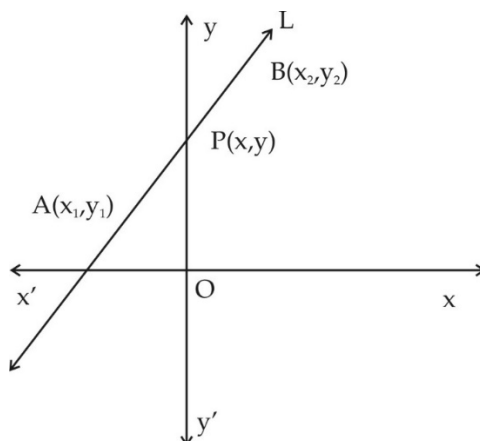
Equation of a Line in Point-Slope Form

Let L be a non-vertical line with slope m and passing through a point A (x_1, y_1).

Let P (x, y) be an arbitrary point on L.

Then slope of the line L = $\dfrac{y - y_1}{x - x_1}$

$\therefore$ m = $\dfrac{y - y_1}{x - x_1}$ $\Rightarrow$ y–y_1 = m (x – x_1) where m = tan θ
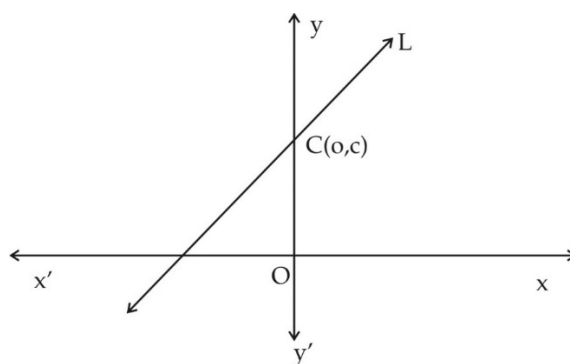
Equation of a Line in Two-Point Form

Let L be the line passing through the points A (x_1, y_1) and B (x_2, y_2) and let P (x, y) be an arbitrary point on L. Then slope of AP = slope of AB.

$$\Rightarrow \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

which is the required equation.

Equation of a Line in Slope-Intercept Form

Let L be a given line with slope m, making an intercept c on the y-axis. Then, L cuts the y-axis at the point (O,C).
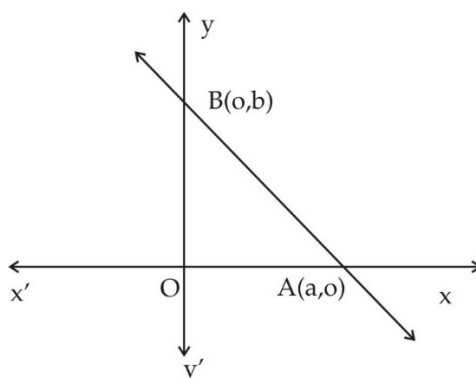
(Remember C could be –ve, O or +ve. The y intercept of the line is C.)

Thus, the line L has slope m and it passes through the point C (o, c).

So by point-slope form, the equation of L is

$$y - c = m(x - 0) \Rightarrow y = mx + c$$

Equation of a Line in Intercept Form

Let AB be the line cutting the x-axis and the y-axis at A (a, o) and B (o, b) respectively. Here, x – intercept and y – intercept of the line are a, b respectively. Let us assume that none of a, b is 0.

So, the equation of line AB is given by

$$\frac{y-0}{x-a} = \frac{b-0}{0-a}$$

$$\Rightarrow \frac{y}{x-a} = \frac{b}{-a}$$
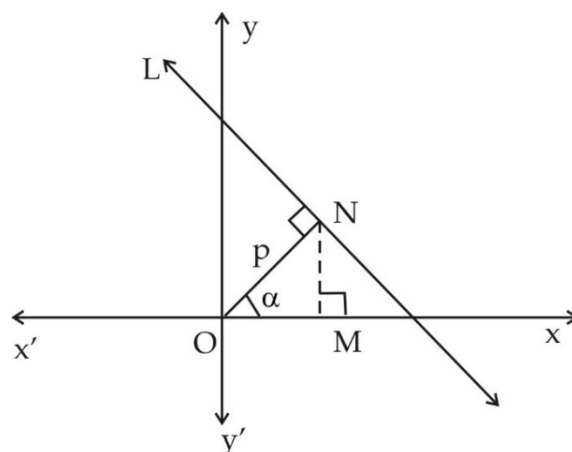
$$\Rightarrow b\,(x-a) = -\,ay$$

$$\Rightarrow bx + ay = ab$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1$$

(Division of both sides by ab is possible as none of a and b is 0)

Thus, the equation of a line making intercepts a and b on the x and y axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$.

Equation of a Line in Normal Form

Let L be the given line. Draw ON perpendicular to L. Let ON = p and let the angle between positive direction of the x-axis and ON be α in anti-clockwise sense.

Draw perpendicular NM on the x-axis. We have

$$\frac{OM}{ON} = \cos \alpha \text{ and } \frac{NM}{ON} = \sin \alpha$$

(In this figure α is acute. But, α could be such that $0 \le \alpha < 2\pi$) In all possible cases, the co-ordinates of N are (p cos α, sin α)

$(\alpha \ne \dfrac{\pi}{2}, \alpha \ne \dfrac{3\pi}{2})$

$\Rightarrow$ OM = p cos α and NM = p sin α

Thus the coordinates of N are (p cos α, p sin α)

Moreover, line L is perpendicular to ON.

$\therefore$ Slope of the line L = $\dfrac{-1}{\text{Slope of ON}} = \dfrac{-1}{\tan \alpha} = \dfrac{-\cos \alpha}{\sin \alpha}$

Thus, the line L passes through the point N (p cos α, p sin α) and has slope

$$m = -\frac{\cos \alpha}{\sin \alpha}$$

So, the equation of the line L is given by

$$y - p \sin \alpha = -\frac{\cos \alpha}{\sin \alpha} (x - p \cos \alpha)$$

$\Rightarrow$ (y – p sin α) sin α = – cos α (x – p cos α)

$\Rightarrow$ x cos α + y sin α = p (cos^2 α + sin^2 α)

$\Rightarrow$ x cos α + y sin α = p

which is the required equation.

Theorem: Prove that the equation Ax + By + C = 0 always represents a line provided A and B are not simultaneously zero.

Proof: Let Ax + By + C = 0 $\qquad\qquad$ where A^2 + B^2 $\ne$ 0

Case I: When A = 0 and B $\ne$ 0

In this case, the given equation becomes

$$By + C = 0 \Rightarrow y = -\frac{C}{B}$$

which represents a line parallel to the x-axis

Case II: When $A \neq 0$ and $B = 0$

In this case, the given equation becomes

$$Ax + C = 0 \Rightarrow x = -\frac{C}{A}$$

which represents a line parallel to the y-axis

You may note that even if $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, the equation would be of the form $y = p$ or

$-y = p$ which is of the form $x \cos = \frac{\pi}{2} + y \sin \frac{\pi}{2} = p$ or $x \cos \frac{3\pi}{2} + y \sin \frac{3\pi}{2} = p$

Case III: When $A \neq 0$ and $B \neq 0$

In this case we have

$$Ax + By + C = 0$$

$$\Rightarrow By = -Ax + (-C)$$

$$\Rightarrow y = \left(-\frac{A}{B}\right) x + \left(-\frac{C}{B}\right)$$

This is clearly an equation of a line in slope intercept form $y = mx + c$,

where $m = \left(-\frac{A}{B}\right)$ and $\left(-\frac{C}{B}\right)$

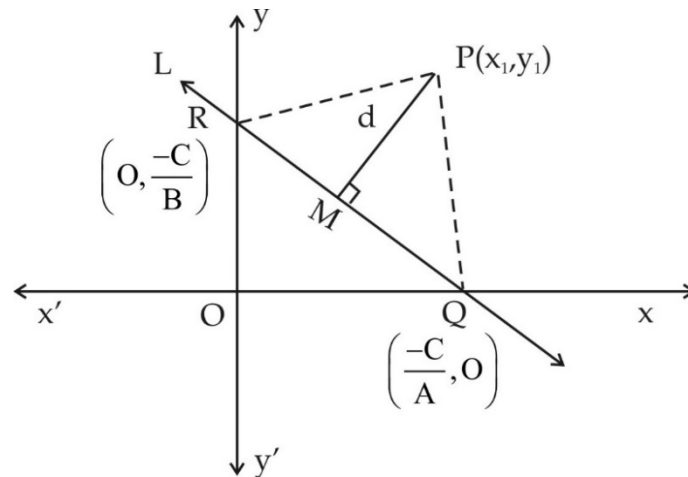Thus $Ax + By + C = 0$ where $A^2 + B^2 \neq 0$ always represents a line.

Theorem: Prove that the length of perpendicular from a given point P (x_1, y_1) on a line $Ax + By + C = 0$ is given by

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Proof: Let L be the line represented by the equation $Ax + By + C = 0$ and let P (x_1, y_1) be a given point outside it.

Let us assume that none of A, B, C is 0

Let this line L intersect the x-axis and the y-axis at points Q and R respectively, and let PM ⊥ QR.

Now, Ax + By + C = 0 ⇒ Ax + By = –C

$$\Rightarrow \frac{Ax}{-C} + \frac{By}{-C} = 1$$

$$\Rightarrow \frac{x}{\left(-\dfrac{C}{A}\right)} + \frac{y}{\left(-\dfrac{C}{B}\right)} = 1$$

Thus, the given line intersects the x-axis and the y-axis at the points $Q\left(-\dfrac{C}{A}, 0\right)$ and $R\left(0, -\dfrac{C}{B}\right)$ respectively.

Area of Δ PQR = $\dfrac{1}{2}\left| x_1\left(\dfrac{-C}{B} - 0\right) + 0.(0 - y_1) + \left(\dfrac{-C}{A}\right)\left(y_1 + \dfrac{C}{B}\right)\right|$

$$= \frac{1}{2}\left| -C\left(\frac{x_1}{B} + \frac{y_1}{A} + \frac{C}{AB}\right)\right|$$

$$= \frac{1}{2}\left| \frac{C}{AB}(Ax_1 + By_1 + C)\right|$$

$$= \left| \frac{C}{2AB}(Ax_1 + By_1 + C)\right| \qquad (1)$$

(13)

Also, area of $\triangle$ PQR = $\dfrac{1}{2}$ × QR × PM

$$= \dfrac{1}{2} \times PM \times \sqrt{(OQ)^2 + (OR)^2}$$

$$= \dfrac{1}{2} \times PM \times \sqrt{\left(-\dfrac{C}{A}\right)^2 + \left(-\dfrac{C}{B}\right)^2}$$

$$= \dfrac{1}{2} \times PM \times \sqrt{\dfrac{C^2}{A^2} + \dfrac{C^2}{B^2}} \qquad\qquad (2)$$
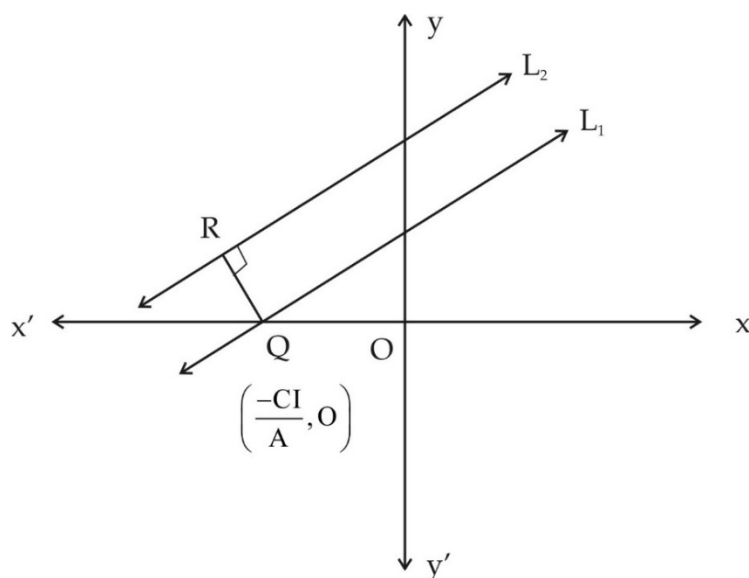
From equation (1) and (2), we get

$$\dfrac{1}{2} \times PM \times \sqrt{\dfrac{C^2}{A^2} + \dfrac{C^2}{B^2}} = \left|\dfrac{C}{2AB}\left(Ax_1 + By_1 + C\right)\right|$$

$$\Rightarrow \dfrac{1}{2} \times \left|\dfrac{C}{AB}\right| \times \sqrt{A^2 + B^2} \times PM = \left|\dfrac{C}{2AB}\right|\left|\left(Ax_1 + By_1 + C\right)\right|$$

$$\Rightarrow PM = \dfrac{\left|Ax_1 + By_1 + C\right|}{\sqrt{A^2 + B^2}}$$

Hence, d = $\dfrac{\left|Ax_1 + By_1 + C\right|}{\sqrt{A^2 + B^2}}$

It may be easily seen that the formula stands true even if A = 0, B ≠ 0, C = 0 or A ≠ 0, B = 0, C = 0 or A = 0, B ≠ 0, C ≠ 0 or A ≠ 0, B = 0, C ≠ 0.

Remarks: The length of perpendicular from the origin (0, 0) on the line Ax + By + C = 0 is $\dfrac{|C|}{\sqrt{A^2 + B^2}}$

Distance between Two Parallel Lines

Theorem: Prove that the distance between two parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is given by

$$d = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}$$

Solution: Let L_1 and L_2 be two given parallel lines represented by

$$Ax + By + C_1 = 0 \qquad\qquad (1)$$

and $\quad Ax + By + C_2 = 0 \qquad\qquad (2)$

Putting $y = 0$ in equation (1), we get $x = -\dfrac{C_1}{A}$ (assuming $A \neq 0$)

Thus L_1, intersects the x-axis at $Q = \left(-\dfrac{C_1}{A},\ 0 \right)$

Let $QR \perp L_2$ and let $d = |QR|$. Then

d = length of perpendicular from $Q\left(-\dfrac{C_1}{A},\ 0 \right)$ on $Ax + By + C_2 = 0$

$$\Rightarrow d = \frac{\left| A \times \left(-\dfrac{C_1}{A} \right) + B \times 0 + C_2 \right|}{\sqrt{A^2 + B^2}} = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}$$

(15)

Illustrative Examples

Microeconomic applications: Demand curve

Example 1: Let us assume that the demand curve is described by the line $q = mp + b$. Find its equations given that a promoter discovers that the demand for theatre tickets is 1200 when the price is Rs. 400, but decreases to 900 when the price is raised to Rs. 450.

Solution: The form of the equation $q = mp + b$ indicates that the price p, is the independent variable and the quantity q, is the dependent variable. The problem gives us two points of the demand line as (400, 1200) and (450, 900). We must identify the slope and the y-intercept of the line.

Slope m = $\dfrac{\Delta q}{\Delta p} = \dfrac{q_2 - q_1}{p_2 - p_1} = \dfrac{900 - 1200}{450 - 400} = \dfrac{-300}{50} = -6$

The equation must, therefore, take on the form, $q = -6p + b$.

Now since (400, 1200) is a point on the demand curve it must satisfy the equation $q = -6 p + b$. By substitution, we obtain

$1200 = -6 (400) + b$

or $\quad 1200 = -2400 + b$

$b = 3600$

Now, since m = – 6 and b = 3600, the equation of the demand line is

$q = -6 p + 3600$

Example 2: The equilibrium quantity and the equilibrium price of a product are determined by the point where the supply and demand curves intersect. For a given product, the supply is determined by the line.

q supply = 30 p – 45

and for the same product, the demand is determined by the line

q demand = – 15 p + 855

Determine the equilibrium price and the equilibrium quantity and trace the supply and demand lines on the same graph.

Solution: We must determine the coordinates of point (q, p) situated at the intersection of the two lines. This point must, therefore, satisfy both the supply and the demand equations. The solution to this problem is to solve:

$q = 30\,p - 45$ (1)

$q = -15\,p + 855$ (2)

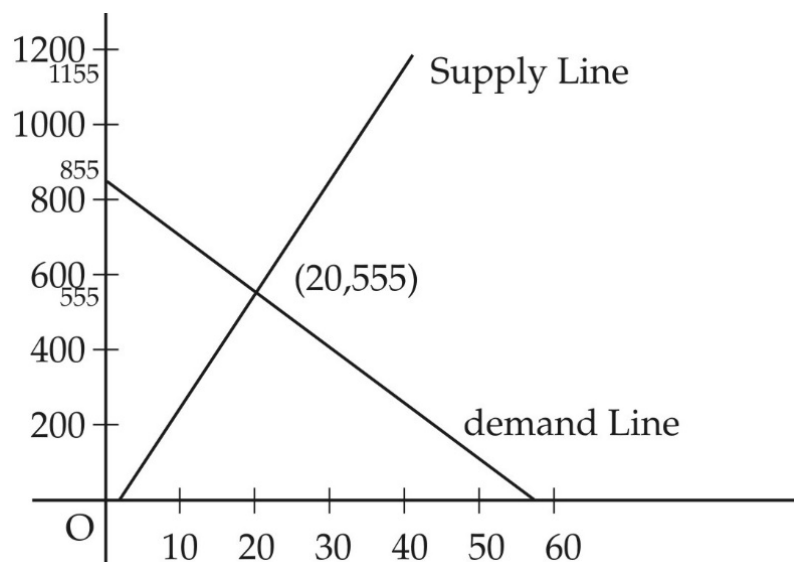Thus from equation (1) and (2)

$30\,p - 45 = -15\,p + 855$

or $45\,p = 900$

       $p = 20$

and $q = 30\,(20) - 45 = 555$

The equilibrium price and the equilibrium quantity are, therefore, Rs. 20 and 555 resepctively.

Example 3: Find the equations of the lines parallel to the axes and passing through the point (–3, 5).

Solution: We know that
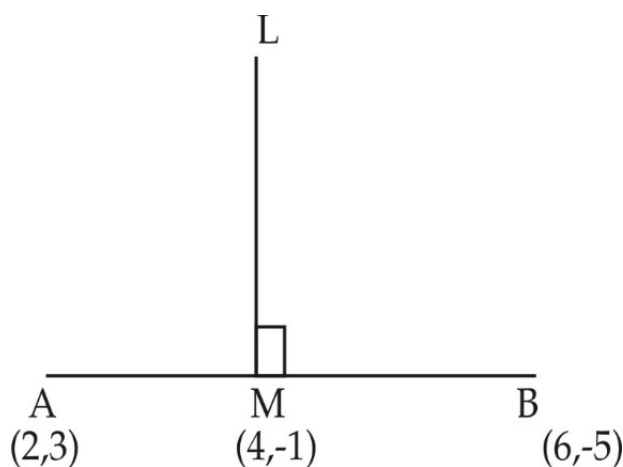
(i) The equation of a line parallel to the x-axis and passing through (–3, 5) is
y = 5

(ii) The equation of a line parallel to the y-axis and passing through (–3, 5) is
x = –3

Example 4: Find the equation of the perpendicular bisector of the line segment joining the points A (2, 3) and B (6, –5)

Solution: Here A (2, 3) and B (6, –5) are the end points of the given line segment. The midpoint of AB is

$M\left(\dfrac{2+6}{2}, \dfrac{3-5}{2}\right)$ i.e., M (4, –1)

Slope of AB = $\dfrac{-5-3}{6-2}$ = –2

$$\begin{array}{ccc} A & M & B \\ (2,3) & (4,-1) & (6,-5) \end{array}$$

Let $LM \perp AB$ and let the slope of LM be m.

Thus, mx (–2) = –1 $\Rightarrow$ m = $\dfrac{1}{2}$

Now, LM is the perpendicular bisector of AB with slope = $\dfrac{1}{2}$ and passing through M (4, –1).

So, the required equation is

$$y + 1 = \frac{1}{2}(x - 4)$$

or $2y + 2 = x - 4$

$\Rightarrow$ $x - 2y - 6 = 0$

Example 5: Find the equation of the line whose y-intercept is –3 and which is perpendicular to the line $3x - 2y + 5 = 0$.

Solution: $3x - 2y + 5 = 0 \Rightarrow 2y = 3x + 5$

$$\Rightarrow y = \frac{3}{2}x + \frac{5}{2}$$

$\therefore$ Slope of the given line $= \dfrac{3}{2}$

$\therefore$ Slope of a line perpendicular to the given line

$$= -\frac{2}{3} \qquad\qquad (\because m_2 = -\frac{1}{m_1})$$
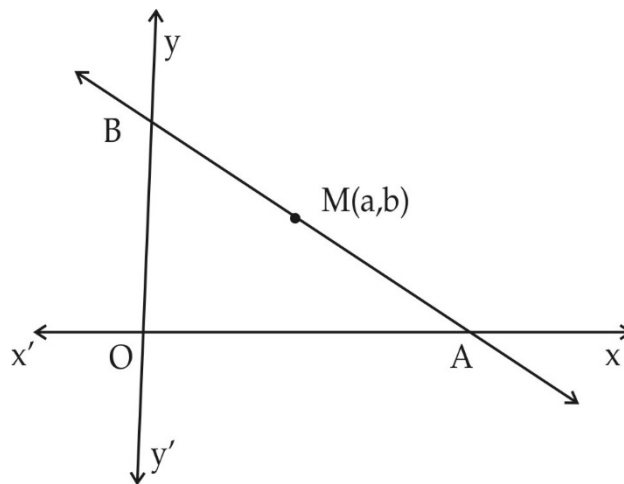
Now, the equation of a line with slope $-\dfrac{2}{3}$ and y-intercept –3 is given by

$$y = -\frac{2}{3}x - 3$$

$\Rightarrow 2x + 3y + 9 = 0$ which is the required equation.

Example 6: If M (a, b) is the midpoint of a line segment intercepted between the axes, show that the equation of the line is $\dfrac{x}{a} + \dfrac{y}{b} = 2$.

Solution: Let the equation of the line be $\dfrac{x}{c} + \dfrac{y}{d} = 1$

Thus, it cuts the x-axis and y-axis at the points A (c, 0) and B (o, d). M (a, b) is the midpoint of AB. So $\dfrac{c+0}{2}$ = a and $\dfrac{0+d}{2}$ = b.

$\Rightarrow$ c = 2a and d = 2b.

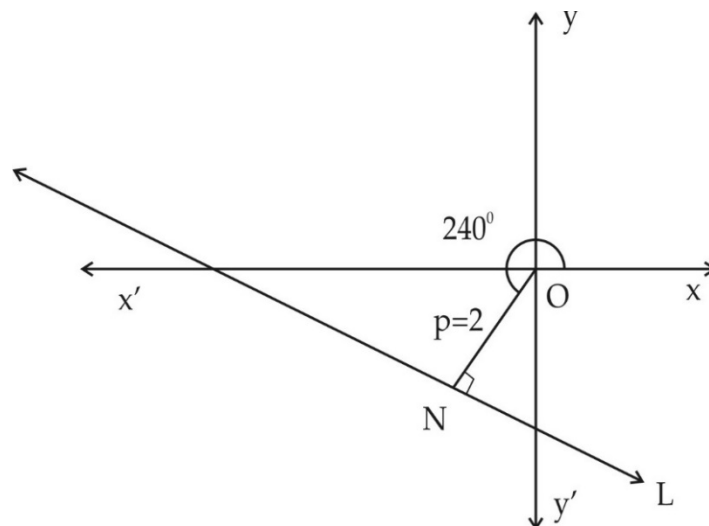So the required equation is

$$\frac{x}{2a}+\frac{y}{2b}=1 \quad \Rightarrow \quad \frac{x}{a}+\frac{y}{b}=2$$

Example 7: Find the equation of a line whose perpendicular distance from the origin is 2 units and the angle between the perpendicular segment and the positive direction of the x-axis is 240^0.

Solution: Let equation of the line be x cos α + y sin α = p
Here p = 2 units and α = 240^0

$$\cos 240^0 = \cos(180^0 + 60^0) = -\cos 60^0$$

$$= -\frac{1}{2}$$

$$\sin 240^0 = \sin(180^0 + 60^0) = -\sin 60^0$$

$$= \frac{-\sqrt{3}}{2}$$

So, the equation of the line is

$$x \cos 240^0 + y \sin 240^0 = 2$$

$$\Rightarrow x \times \left(-\frac{1}{2}\right) + y \times \left(-\frac{\sqrt{3}}{2}\right) = 2$$

$$\Rightarrow x + \sqrt{3}\,y + 4 = 0$$

which is the required equation.

Example 8: Reduce the equation $\sqrt{3}\,x + y + 2 = 0$ to the normal form $x \cos \alpha + y \sin \alpha = p$ and hence find the value of α and p.

Solution: We have

$$\sqrt{3}\,x + y + 2 = 0 \Rightarrow -\sqrt{3}x - y = 2$$

$$\left(-\frac{\sqrt{3}}{2}\right)x + \left(-\frac{1}{2}\right)y = 1$$

Normal form of the equation is $x \cos \alpha + y \sin \alpha = p$

Here, $\cos \alpha = -\frac{\sqrt{3}}{2}$, $\sin \alpha = -\frac{1}{2}$, $p = 1$

Since $\cos \alpha < 0$ and $\sin \alpha < 0$, α lies in the third quadrant.

Now, $\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \left(-\frac{1}{2}\right) \times \left(-\frac{2}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} = \tan(180^0 + 30^0)$

$$= \tan 210^0 \Rightarrow \alpha = 210^0$$

Thus, $\alpha = 210^0$ and $p = 1$

Hence, the given equation in normal form is $x \cos 210^0 + y \sin 210^0 = 1$.

Example 9: Find the distance between the parallel lines.

$$15x + 8y - 34 = 0 \text{ and } 15x + 8y + 31 = 0$$

Solution: Converting each of the given equation to the form $y = mx + c$, we get

$$15x + 8y - 34 = 0 \Rightarrow y = -\frac{15}{8}x + \frac{17}{4} \qquad (1)$$

$$15x + 8y + 31 = 0 \Rightarrow y = -\frac{15}{8}x - \frac{31}{8} \qquad (2)$$

Clearly, the slopes of the given lines are equal and so they are parallel.

The given lines are of the form $y = mx + c_1$ and $y = mx + c_2$ where $m = -\frac{15}{8}$, $c_1 = \frac{17}{4}$ and $c_2 = -\frac{31}{8}$

$\therefore$ Distance between the given lines

$$= \frac{|c_2 - c_1|}{\sqrt{1 + m^2}}$$

$$= \frac{\left|-\frac{31}{8} - \frac{17}{4}\right|}{\sqrt{1 + \left(-\frac{15}{8}\right)^2}} = \frac{\left|-\frac{65}{8}\right|}{\sqrt{1 + \frac{225}{64}}} = \frac{\frac{65}{8}}{\sqrt{\frac{289}{64}}}$$

$$= \frac{65}{8} \times \frac{8}{17} = \frac{65}{17} \text{ units}$$

Hence, the distance between the given lines is $\frac{65}{17}$ units.

Exercise 1

1. Find the equation of a line which is equidistant from the lines y = 8 and y = –2.

2. If A (1, 4), B (2, –3) and C (–1, –2) are the vertices of a ΔABC, then find the equation of

    (i)   the median through A

    (ii) the altitude through A

    (iii) the perpendicular bisector of BC

3. Find the equation of the bisector of the angle between the coordinate axes.

4. Find the equation of the line passing through the point (2, 2) and cutting off intercepts on the axes, whose sum is 9.

5. Find the equation of the line which is at a distance of 3 units from the origin such that $\tan \alpha = \dfrac{5}{12}$, where α is the acute angle which this perpendicular makes with the positive direction of the x-axis.

6. Reduce the equation 5x – 12y = 60 to intercept form. Hence find the length of the portion of the line intercepted between the axes.

7. Reduce the equation x + y – 2 = 0 to the normal form.

8. What are the points on the x-axis whose perpendicular distance from the line $\dfrac{x}{3} + \dfrac{y}{4} = 1$ is 4 units.

9. A company produces shoes. When 30 shoes are produced the total cost of production is Rs. 1500. When 50 shoes are produced the costs increases to Rs. 2000. What is the cost equation (C) if it varies linearly in function to the number of shoes produced (q).

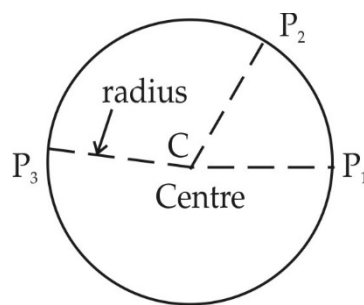10. Consider a market characterised by the following supply and demand curves:

(23)

q demand = – 10p + 1000

q supply = 0.2 p + 2986

Find the equilibrium price and quantity.

12.5 Circle

A circle is the set of all points in a plane, each of which is at a constant distance from a fixed point in the plane.

The fixed point is called the centre and the constant distance is called the radius. Radius is always positive.

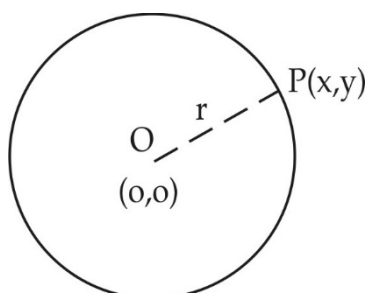It P_1, P_2, P_3, ... are points on the circle with center C, and radius r, then

$$CP_1 = CP_2 = CP_3 = ... = r$$

Equation of a Circle in Standard Form

Let O (0, 0) be the centre of the circle and r be its radius. Let P (x, y) be a point in the plane, then P lies on the circle iff OP = r
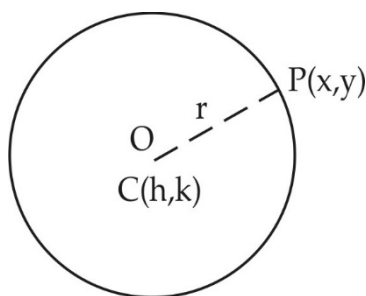
i.e., $\sqrt{(x-0)^2 + (y-0)^2} = r$

$\Rightarrow x^2 + y^2 = r^2$

This is standard form of the equation of a circle.

Equation of a Circle in Central Form

Let C (h, k) be the centre of the circle and r be its radius. Let P (x, y) be a point in the plane, then P lies on the circle if CP = r

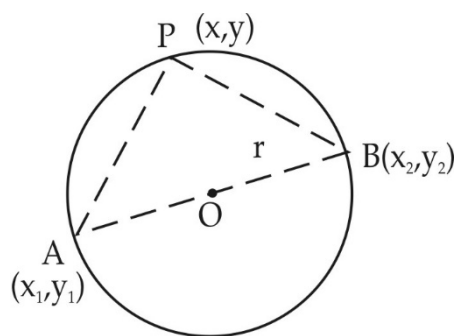

$\Rightarrow \sqrt{(x-h)^2 + (y-k)^2}$ = r

$\Rightarrow (x - h)^2 + (y - k)^2 = r^2$

This is known as the central form of the equation of a circle.

Equation of a Circle in Diameter Form

Let A (x_1, y_1) and B (x_2, y_2) be the extremities of a diameter of the circle.

Let P (x, y), be any point on the circle. Thus

slope of the line AP = $\dfrac{y - y_1}{x - x_1}$ and

slope of the line BP = $\dfrac{y - y_2}{x - x_2}$

Now P lies on the circle, so $\angle APB = 90^0$

The lines i.e. AP and BP are perpendicular to each other.

So $\quad \dfrac{y-y_1}{x-x_1} \cdot \dfrac{y-y_2}{x-x_2} = -1$

$$\Rightarrow (y-y_1)\,(y-y_2) = -(x-x_1)\,(x-x_2)$$

$$\Rightarrow (x-x_1)\,(x-x_2) + (y-y_1)\,(y-y_2) = 0$$

This is the equation of a circle in diameter form.

Equation of a circle in General Form

We know that the equation of the circle with centre (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2$$

$$\Rightarrow x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$$

It can be written is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

where $g = -h$, $f = -k$ and $c = h^2 + k^2 - r^2$

$\therefore \quad r^2 = h^2 + k^2 - c$

$$= (-g)^2 + (-f)^2 - c$$

$$= g^2 + f^2 - c$$

But $r^2 > 0$, so $g^2 + f^2 - c > 0$

Now if we consider any equation

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

with $g^2 + f^2 - c > 0$ then on adding $g^2 + f^2$ on both the sides, we get

$$(x^2 + 2gx + g^2) + (y^2 + 2fg + f^2) + c = g^2 + f^2$$

$\Rightarrow \quad (x+g)^2 + (y+f)^2 = g^2 + f^2 - c$

$\Rightarrow \quad \{x-(-g)\}^2 + \{y-(-f)\}^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$

$\Rightarrow \quad \sqrt{\{x-(-g)\}^2 + \{y-(-f)\}^2} = \sqrt{g^2 + f^2 - c}$

The distance of the point (x, y) from the point $(-g, -f)$ is a fixed positive real number $r \left(= \sqrt{g^2 + f^2 - c} \right)$

This is the set of all points (x, y) which is at a constant distance $r \left(= \sqrt{g^2 + f^2 - c} \right)$ from the fixed point $(-g, -f)$

So the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$. This is general form of the equation of a circle.

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

12.6 Properties of Circles

1.   If $g^2 + f^2 - c = 0$, the equation of circle is satisfied by one and only one point $(-g, -f)$. Therefore it represents a single point known as point circle.

2.   If $g^2 + f^2 - c < 0$, the equation of the circle is not satisfied by any real value of x, y i.e. it is not satisfied by the co-ordinates of any point in the plane.

3.   The general equation of a circle has the following character stics.

   (i)   It is an equation of second degree in x, y containing no product term xy.

   (ii)   Coefficients of x^2 = Coefficients of y^2 = 1

4.   The equation $ax^2 + ay^2 + 2gx + 2fy + c = 0$ represents a circle if $a \neq 0$ and $g^2 + f^2 - c > 0$. Its centre is $\left(-\dfrac{g}{a}, -\dfrac{f}{a} \right)$ and radius is $\dfrac{\sqrt{g^2 + f^2 - ac}}{|a|}$.

   The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle if

   (i) $a = b \neq 0$

   (ii) $h = 0$ and

   (iii) $g^2 + f^2 - ac > 0$

Illustrative Examples

Example 1: Find the equation of a circle with centre (3, –2) and radius 5.

Solution: We know that the equation of a circle with centre C (h, k) and radius r is given by

$$(x – h)^2 + (y – k)^2 = r^2$$

Here h = 3, k = –2 and r = 5

∴ Required equation of the circle is

$$(x – 3)^2 + (y + 2)^2 = 5^2$$

$$\Rightarrow x^2 + y^2 – 6x + 4y – 12 = 0$$

Example 2: Find the equation of the circle with centre (2, 2) and which passes through the point (4, 5).

Solution: The centre of the circle is C (2, 2) and it passes through the point P (4, 5)

∴ radius of circle = CP = $\sqrt{(4-2)^2 + (5-2)^2}$

$$= \sqrt{4+9}$$

$$= \sqrt{13}$$

∴ Equation of the circle is

$$(x – 2)^2 + (y – 2)^2 = \left(\sqrt{13}\right)^2$$

$$\Rightarrow x^2 + y^2 – 4x – 4y – 5 = 0$$

Example 3: Find the equation of the circle of radius 5 whose centre lies on x–axis and passes through the point (2, 3).

Solution: Let equation of the circle be $(x - h)^2 + (y - k)^2 = r^2$ centre lies on x-axis so centre is (h,o) Since the circle passes through A (2, 3) and has radius 5.

∴ $(2 – h)^2 + (3 – 0)^2 = 5^2$

$$\Rightarrow (2-h)^2 = 16 \Rightarrow 2-h = 4, -4$$

$$\Rightarrow h = -2, 6$$

∴ The centre of the circle is (–2, 0) or (6, 0).

Equation of the circle is

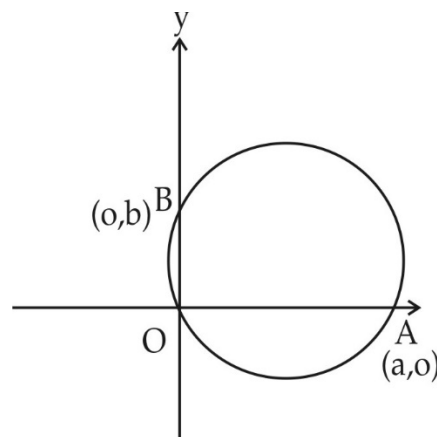$$(x+2)^2 + (y-0)^2 = 5^2 \quad \text{or} \quad (x-6)^2 + (y-0)^2 = 5^2$$

i.e. $\quad x^2 + y^2 + 4x - 21 = 0 \quad$ or $\quad x^2 + y^2 - 12x + 11 = 0$

There are two circles satisfying the given conditions.

Example 4: Find the equation of the circle passing through (0, 0) and which makes intercepts a and b on the coordinate axes.

Solution: Since the circle cuts off intercepts a and b on the coordinate axes, so it passes through the points A (a, 0) and B (o, b).

As the axes are perpendicular to each other and hence AB becomes a diameter of this circle.

∴ The equation of the circle is $(x - a)(x - 0) + (y - 0)(y - b) = 0$

or, $\quad x^2 + y^2 - ax - by = 0$

Example 5: Find the centre and radius of the circle $x^2 + y^2 - 8x + 10y - 12 = 0$.

Solution: The given equation is

$x^2 + y^2 - 8x + 10y - 12 = 0$

Comparing it with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$g = -4, f = 5$ and $c = -12$

$\therefore \; r^2 = g^2 + f^2 - c = (-4)^2 + 5^2 - (-12) = 53 > 0$

Or, $r = \sqrt{53}$

Hence, the given equation represents a circle with centre $(-g, -f)$ i.e., $(4, -5)$
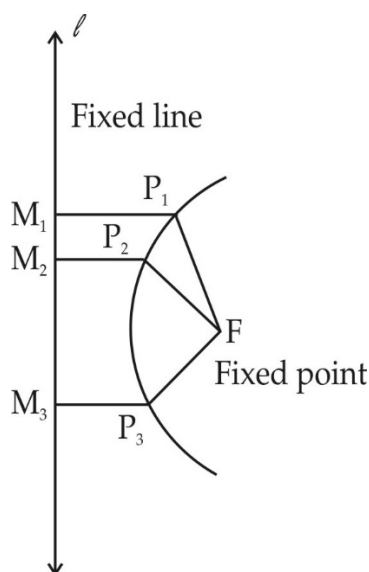
and radius = $\sqrt{53}$

Exercise 2

Very short answer type questions

1. Find the equation of the circle with:

    (i)   centre $(0, 2)$ and radius 2.

    (ii)   centre $(0, 0)$ and radius 3.

    (iii)   centre $(-a, -b)$ and radius $\sqrt{a^2 - b^2}$.

2. Find the equation of the circle drawn on a diagonal of the rectangle as its diameter whose sides are the lines $x = 4, x = -5, y = 5$ and $y = -1$.

3. Which of the following equations represent a circle? If so, determine its centre and radius.

    (i) $3x^2 + 3y^2 + 6x - 4y = 1$

    (ii) $2x^2 + 2y^2 + 3y + 10 = 0$

    (iii) $x^2 + y^2 - 12x + 6y + 45 = 0$

4. One end of a diameter of the circle $x^2 + y^2 - 6x + 5y - 7 = 0$ is $(-1, 3)$. Find the coordinates of the other end of the diameter

5. Find the equation of the circle passing through the points (2, 3) and (–1, 1) and whose centre lies on the line $x - 3y - 11 = 0$.

6. Find the value of p so that $x^2 + y^2 + 8x + 10y + p = 0$ is the equation of a circle of radius 7 units.

7. Find the value of k for which the circles $x^2 + y^2 - 3x + ky - 5 = 0$ and $4x^2 + 4y^2$
   $- 12x - y - 9 = 0$ are concentric.

12.7 Parabola

A parabola is the set of all points in a plane which are equidistant from a fixed line and a fixed point (not on the line) in the plane.

The fixed line (say ℓ) is called the directrix of the parabola and the fixed point (say F) is called the focus of the parabola.
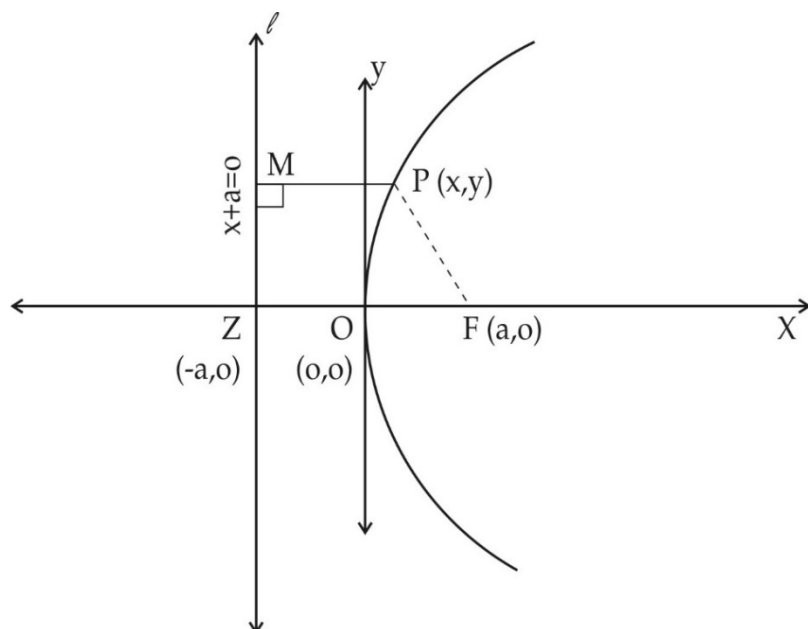
If P_1, P_2, P_3.... are points on the parabola and M_1P_1, M_2P_2, M_3P_3 are perpendiculars to the directrix ℓ, then

$FP_1 = M_1P_1$, $FP_2 = M_2P_2$, $FP_3 = M_3P_3$ etc.

The line passing through the focus and perpendicular to the directrix is called the axis of the parabola. The point of intersection of parabola with the axis is

called the vertex of the parabola. The equation of a parabola is simplest if its vertex is at the origin and its axis lies along either x-axis or y-axis.

Equation of a parabola in the standard form

Let F be the focus, ℓ be the directrix and Z be the foot of perpendicular from F to the line ℓ.

Take ZF as x-axis Let O be the mid-point of ZF. Take O as the origin. Then the line through 0 and perpendicular to ZF becomes y-axis.

Let ZF = 2a (a > 0) then

ZO = OF = a

Since F lies to the right of O and Z to the left of O, coordinates of F and Z are (a, 0) and (–a, 0) respectively. Therefore, the equation of the line ℓ i.e., directrix is x = –a or x + a = 0.

Let P (x,y) be any point in the plane of the line ℓ and the point F, and MP be the perpendicular distance from P to the line ℓ then P lies on the parabola iff

FP = MP

or $\sqrt{(x-a)^2 + y^2} = \dfrac{|x+a|}{1}$

or $\quad (x - a)^2 + y^2 = (x + a)^2$

$\Rightarrow \quad x^2 + a^2 - 2ax + y^2 = x^2 + a^2 + 2ax$

$\Rightarrow \quad y^2 = 4ax$

Hence, the equation of a parabola is $y^2 = 4ax$, $a > 0$, with the focus F (a, 0) and directrix $x + a = 0$

It is called standard form of the equation of a parabola.

Some observations about the parabola $y^2 = 4ax$, $a > 0$

If (x, y) is a point on the parabola $y^2 = 4ax$, then (x, –y) is also a point on the parabola

So, the given parabola is symmetrical about x-axis.

(i)   This line is the axis of symmetry of the parabola $y^2 = 4ax$.

(ii)   The focus is F (a, 0) and directrix is $x + a = 0$.

(iii)   The point O (0, 0) where the axis of parabola meets the parabola is the vertex of the parabola.

(iv)   If x < 0, then $y^2 = 4ax$ has no real solutions in y and so there is no point on the curve with negative x coordinate i.e., on the left of y-axis.
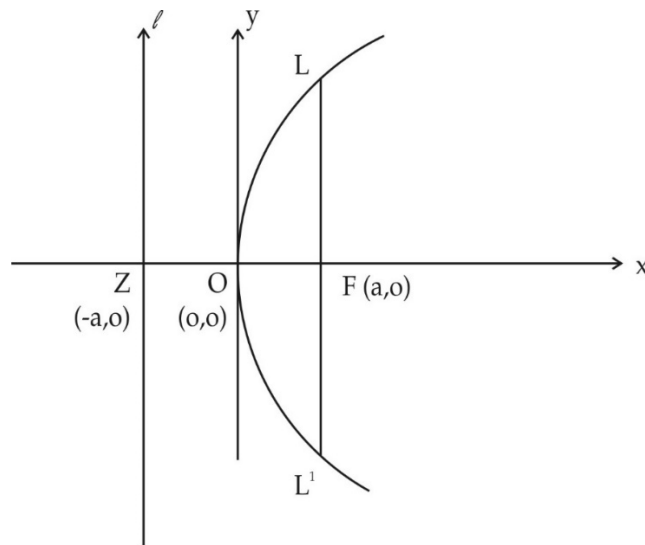
When x = 0 we get $y^2 = 0 \Rightarrow y = 0$. Thus (0, 0) is the only point of the y-axis which lies on it. Therefore the entire curve, except the origin, lies to the right of y-axis.

(v)   A chord passing through the focus F and perpendicular to the axis of the parabola is called the latus rectum of the parabola.

12.8   Length of the Latus Rectum

Let chord LL′ be the latus rectum of the parabola, then LL′ passes through focus F (a, 0) and is perpendicular to x-axis.

Let LF = k (k>0), then the points L and L′ are (a, k) and (a, –k) respectively.

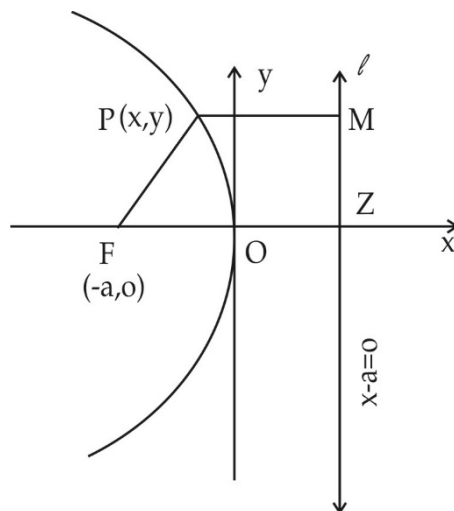As L (a, k) lies on the parabola $y^2 = 4ax$, we get

$$k^2 = 4a \times a \Rightarrow k = 2a$$

∴   The points L and L′ are (a, 2a), (a, −2a) and length of the latus rectum = LL′= 2k = 4a. Also equation of the latus rectum is x = a or x − a = 0.

To find the other equations of a parabola in standard form

(i)   Focus F (−a, 0), a > 0 and the line x − a = 0 as directrix.

(ii)   Focus F (0, a), a > 0 and the line y + a = 0 as directrix.

(iii)   Focus (0, −a), a > 0 and the line y − a = 0 as directrix.

(i)   Let P (x, y) be any point in the plane of directrix and focus and MP be the perpendicular distance from P to the directrix.

Then P lies on parabola.

Iff FP = MP

$$\Rightarrow \sqrt{(x+a)^2 + y^2} = |x - a|$$
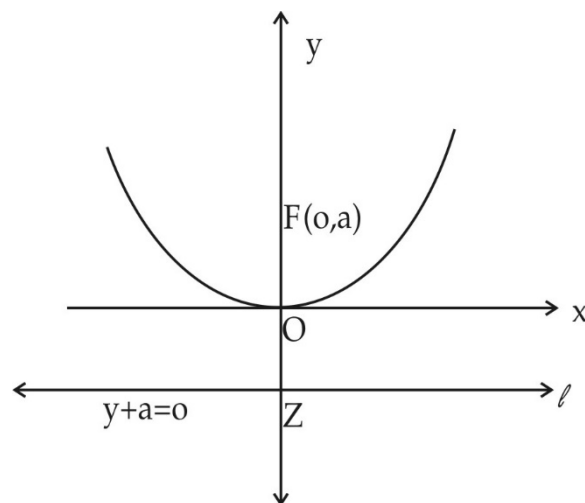
$$\Rightarrow (x + a)^2 + y^2 = (x - a)^2$$

$$\Rightarrow x^2 + a^2 + 2ax + y^2 = x^2 + a^2 - 2ax$$

$$\Rightarrow y^2 = -4ax, a > 0$$

The axis is $y = 0$, equation of directrix $x - a = 0$, focus $(-a, 0)$. Length of latus rectum = $4a$, equation of latus rectum $x + a = 0$.

(ii)   The equation of the parabola opening upwards is $x^2 = 4ay$, $a > 0$.
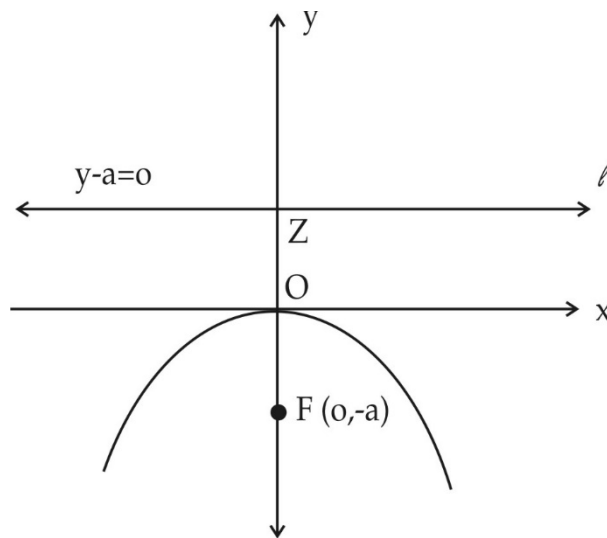
Axis $x = 0$, directrix $y + a = 0$, Focus $(0, a)$.

Length of the latus rectum = $4a$

Equation of latus rectum is $y - a = 0$

(iii)   The equation of the parabola opening downwards is

$$x^2 = -4 \, ay, a > 0$$
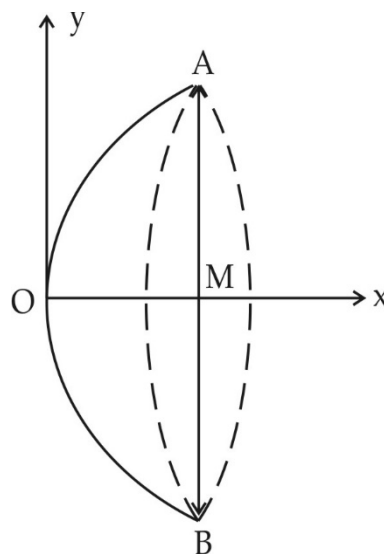
Axis $x = 0$, directrix $y - a = 0$, Focus $(0, -a)$.

Length of the latus rectum = 4a

Equation of the latus rectum is $y + a = 0$

Illustrative Examples

Example 1: If a parabolic reflector is 20 cm in diameter and 5 cm deep, find the focus.

Solution: Let AOB be the parabolic reflector which is 20 cm in diameter and 5 cm deep, then AB = 20 cm and OM = 5cm, where M is midpoint of AB.
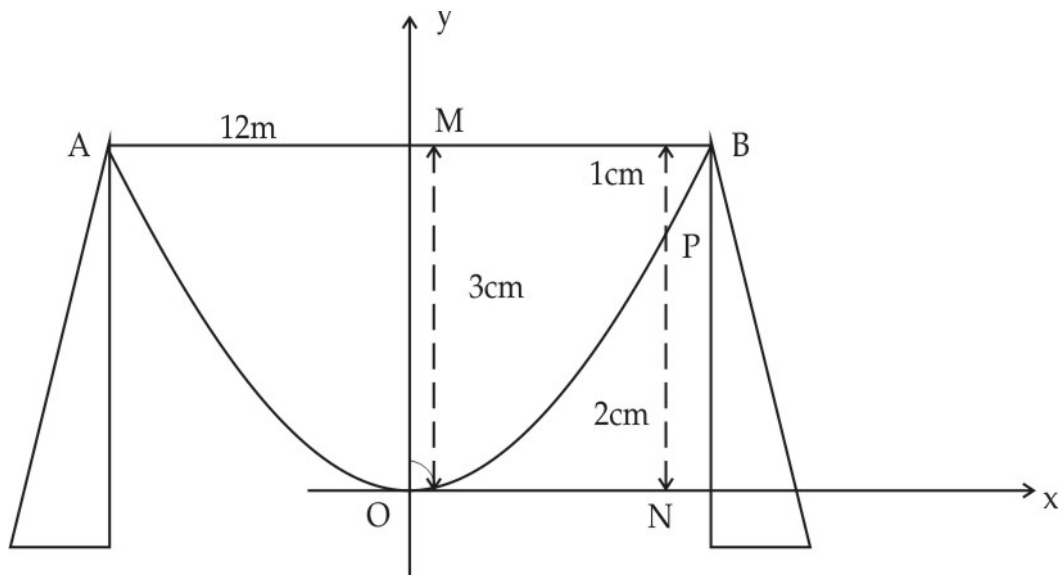

The equation of the parabola can be taken as $y^2 = 4ax$

Since the point A (5, 10) lies on the parabola, so

$$10^2 = 4a \times 5 \Rightarrow a = 5$$

∴ The coordinates of the focus are (a, 0) i.e., (5, 0).

Example 2: A beam is supported at its ends by supports which are 12 metres apart. Since the load is concentrated at the centre of the beam there is a deflection of 3 cm at the centre and the deflected beam is in the shape of a parabola. How far from the centre is the deflection 1 cm?

Solution:

Let AOB be the beam in the shape of a parabola with vertex at the lowest point O. The beam is supported at the points A and B.

Take the vertex O of the parabola as origin and the axis of the parabola as y-axis. The equation of the parabola is $x^2 = 4ay$.

Given AB = 12 m and OM = 3 cm = $\dfrac{3}{100}$ m.

As M is the midpoint of AB, MB = 6 m.

The coordinates of B are $\left(6, \dfrac{3}{100}\right)$

Since B lies on the parabola

$$6^2 = 4a \times \frac{3}{100}$$

$$\Rightarrow a = \frac{36 \times 100}{12} = 300$$

Let P be any point on the parabola whose deflection is 1 cm, then NP = 2 cm = $\frac{2}{100}$ m. Let ON = x metre, then the coordinates of P are $\left(x, \frac{2}{100} \right)$.
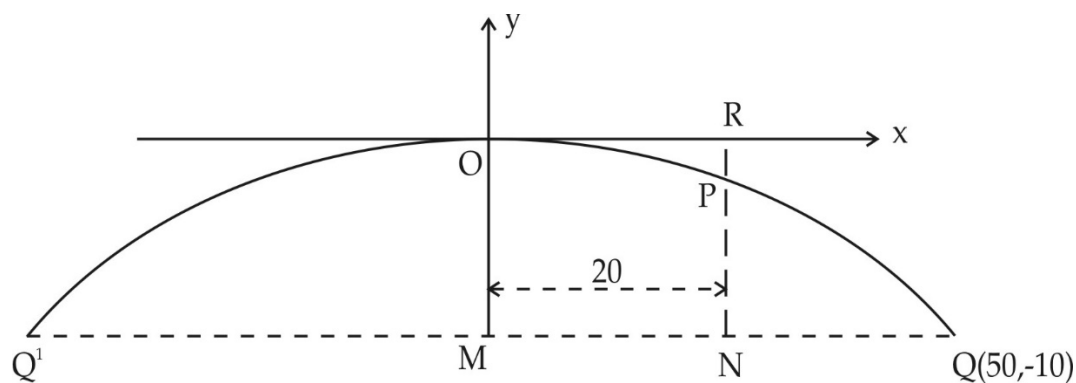
Since P lies on the parabola, we get

$$x^2 = 4 \times 300 \times \frac{2}{100}$$

$$\Rightarrow x^2 = 24 \Rightarrow x = \pm 2\sqrt{6}.$$

Hence, the points of the beam where the deflection is 1 cm are at a distance of $2\sqrt{6}$ m from the centre.

Example 3: The girder of a railway bridge is a parabola with its vertex at the highest point 10 metre above the ends. If the span is 100 metres, find height of the girder at 20 metres from its mid-point.

Solution:

Take the vertex O of the parabola as origin and the axis of the parabola as y-axis. Therefore, the equation of the parabola is $x^2 = -4$ ay.

Given OM = 10 metre and Q'Q = 100 metres

As M is the midpoint of Q'Q, MQ = 50 metres. Therefore coordinates of Q are (50, −10).

Since Q lies on the parabola, so

$$50^2 = -4a(-10) \Rightarrow a = \frac{125}{2}$$

Let MN = 20 metres, draw NR ⊥ MQ to meet the parabola at P.

As P lies below the x-axis, coordinates of P are (20, –p) where p = PR > 0

Since P lies on the parabola, we get

$$20^2 = -4.\frac{125}{2}(-p) \Rightarrow p = \frac{400}{250} = \frac{8}{5} = 1.6$$

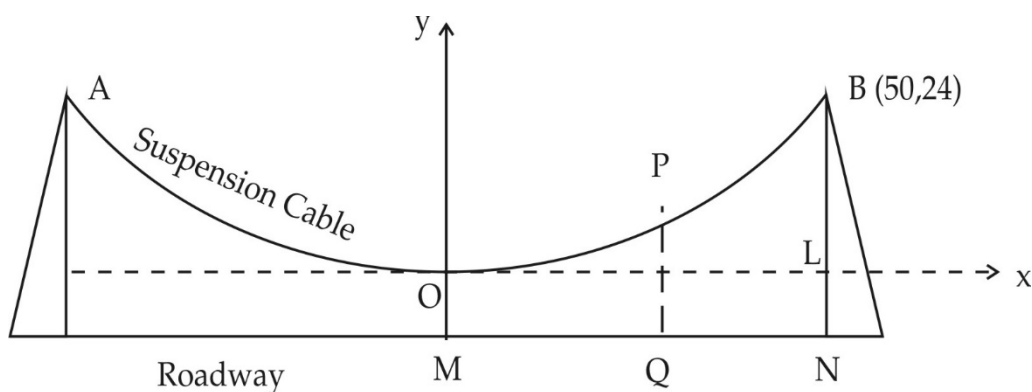∴ The required height = NP = NR – PR

= 10 – 1.6   as NR = OM = 10 m

= 8.4 metres

Example 4: The cable of a uniformly loaded bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of the supporting wire attached to the roadway 18 m from the middle.

Solution:

The bridge is hung by supporting wires in a parabolic arc with vertex at the lowest point and the axis vertical. The equation of the parabola can be taken as

$x^2 = 4$ ay.

Shortest supporting wire = OM = 6 m.

Longest supporting vertical wire = NB = 30 m.

$\therefore$ LB = 30 – 6 = 24 m.

Also OL = $\dfrac{1}{2}$ × 100 m = 50 m.

$\therefore$ The point B is (50, 24) which lies on the parabola, and so we get

$$(50)^2 = 4a \times 24 \Rightarrow a = \frac{2500}{96}$$

Let PQ be the vertical supporting wire at a distance of 18 m from M. If PQ = k metres, then the point P (18, k – 6) lies on parabola, we get
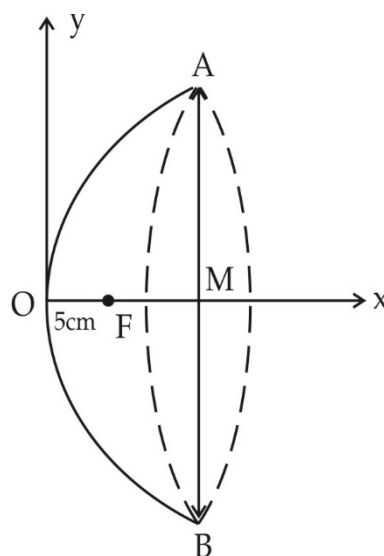
$$18^2 = 4 \times \frac{2500}{96} \ (k-6) \Rightarrow \frac{324 \times 24}{2500} = k - 6$$

$$\Rightarrow k = \frac{324 \times 24}{2500} + 6 = 9.11 \text{ metres (approx.)}$$

Hence the length of the wire required = 9.11 metres (approx.)

Exercise 3

1.   The focus of the parabolic mirror is at a distance of 5 cm from its vertex. If the mirror is 45 cm deep, find the distance AB.

2. An arc is in the form of a parabola with its axis vertical. The arc is 10 m high and 5 m wide at the base. How wide is it 2 m from the vertex of the parabola?

3. The towers of a suspension bridge, hang in the form of a parabola, have their tops 30 metres above the roadway and are 200 metres apart. If the cable is 5 metres above the roadway at the centre of the bridge, find the length of the vertical supporting cable 30 metres from the centre.

4. The girder of a railway bridge is in the form of a parabola with its vertex at the highest point, 15 metres above the ends. If the span is 150 metres, find its height at 30 metres from the midpoint.

5. A water jet from a fountain reaches its maximum height of 4 metres at a distance of 0.5 metres from the vertical passing through the point O of the water outlet. Find the height of the jet above the horizontal OX at a distance 0.75 metre from the point O.

Answer Unit – VIII
Exercise – 1

1. Y = 3

2. (i) $13x - y - 9 = 0$ (ii) $3x - y11 = 0$ (iii) $3x - y + 4 = 0$

3. $y = \pm x$

4. $x + 2y = 6$

5. $12x = 5y = 39$

6. $\dfrac{x}{12} + \dfrac{y}{-5} = 6$, a = 12, b = -5

7. $x\,Cos\,45^0 + y\,Sin45^0 = \sqrt{2}$

8. (8, 0), (-2, 0)

9. $c = 25 + 75^0$

10. 68, 76 and 312, 35

Exercise – 2

1. (i) $x^2 + y^2 - 4y = 0$

   (ii) $x^2 + y^2 - 9 = 0$

   (iii) $x^2 + y^2 + 2ax + 2by + 2b^2 = 0 = 0$

2. $x^2 + y^2 + x - 4y - 25 = 0$

3. (i) Circle, Centre $(-1, \frac{2}{3})$, radius = $\frac{4}{3}$

   (ii) Not circle

   (iii) A point circle with centre (6, -3) and radius zero

4. (7, -8)

5. $x^2 + y^2 + 7x + 5y - 14 = 0$

6. -8

7. $-\dfrac{1}{4}$

Exercise – 3

1. 1.56 m approx.
2. 60 m
3. 2.23 m approx.
4. 7.25 m
5. 12.6 m
6. 13 m